

# Frieze patterns and Farey complexes

Ian Short, Matty Van Son, and Andrei Zabolotskii

## Abstract

Frieze patterns have attracted significant attention recently, motivated by their relationship with cluster algebras. A longstanding open problem has been to provide a combinatorial model for frieze patterns over the ring of integers modulo  $n$  akin to Conway and Coxeter’s celebrated model for positive integer frieze patterns. Here we solve this problem using the Farey complex of the ring of integers modulo  $n$ ; in fact, using more general Farey complexes we provide combinatorial models for frieze patterns over any rings whatsoever.

Our strategy generalises that of the first author and of Morier-Genoud et al. for integers and that of Felikson et al. for Eisenstein integers. We also generalise results of Singerman and Strudwick on diameters of Farey graphs, we recover a theorem of Morier-Genoud on enumerating friezes over finite fields, and we classify those frieze patterns modulo  $n$  that lift to frieze patterns over the integers in terms of the topology of the corresponding Farey complexes.

## 1 Introduction

The purpose of this paper is to provide a combinatorial model for classifying  $SL_2$ -tilings and frieze patterns over any ring. In particular, we solve the outstanding problem of describing such a model for frieze patterns over the ring of integers modulo  $n$  in a manner similar to that used by Conway and Coxeter for positive integer frieze patterns.

Frieze patterns (or, more concisely, friezes) are bi-infinite arrays of numbers of the type shown in Figure 1.1, in which any diamond of four entries  $a$ ,  $b$ ,  $c$ , and  $d$  satisfies the rule  $ad - bc = 1$ .

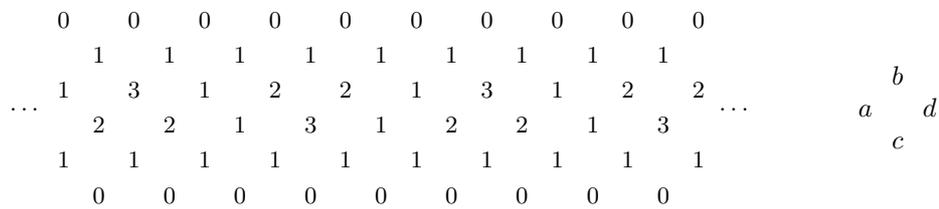


Figure 1.1. A frieze (left) and a diamond of four entries (right)

This frieze has integer entries; however, we will consider friezes with entries in any ring, where we assume, throughout, that all our rings are commutative and contain a multiplicative identity element 1.

---

2010 Mathematics Subject Classification: Primary 05E16; Secondary 11B57.  
 Key words: frieze, Farey complex,  $SL_2$ -tiling.  
 Research supported by EPSRC grants EP/W002817/1 (IS and MvS) and EP/W524098/1 (AZ).  
 There is no data associated with this article.





forms a group under multiplication. A pair of elements  $(a, b)$  in  $R \times R$  is said to be a *unimodular pair* if  $aR + bR = R$  (or, equivalently, if there exist  $x, y \in R$  with  $ax + by = 1$ ). Any subgroup  $U$  of  $R^\times$  acts on the collection of unimodular pairs by the rule  $(a, b) \mapsto (\lambda a, \lambda b)$ , where  $\lambda \in U$ . We write  $\lambda(a, b)$  for  $(\lambda a, \lambda b)$ , and we also use the notation

$$U(a, b) = \{\lambda(a, b) : \lambda \in U\}$$

for the orbit of  $(a, b)$  under  $U$ .

**Definition.** Let  $R$  be a ring and let  $U$  be a group of units in  $R$ . The *Farey complex*  $\mathcal{F}_{R,U}$  of  $R$  with units  $U$  is the directed graph with the following vertices and edges.

- The vertices are orbits  $U(a, b)$ , where  $(a, b)$  is a unimodular pair in  $R \times R$ . We write vertices as formal fractions  $a/b$ .
- There is a directed edge from  $a/b$  to  $c/d$  if  $ad - bc \in U$ .

We represent the directed edge from  $a/b$  to  $c/d$  by  $a/b \rightarrow c/d$ . Observe that there are no directed edges in  $\mathcal{F}_{R,U}$  with the same source and sink, and there is at most one directed edge from one vertex to another.

When  $-1 \in U$ , there is a directed edge from vertex  $u$  to vertex  $v$  if and only if there is a directed edge from  $v$  to  $u$ . Thus, directed edges come in inverse pairs, and we can think of each pair as a single undirected edge. In this case, we define a *face* of  $\mathcal{F}_{R,U}$  to consist of a triple of vertices  $u, v$ , and  $w$  that are adjacent in pairs. The Farey complex  $\mathcal{F}_{R,U}$  is then a 2-complex, and we can examine its geometric properties, as we shall see in due course.

The Farey complexes of most interest to us are those with  $U = \{1\}$ ,  $U = \{\pm 1\}$ , and  $U = R^\times$ . For these three complexes, in place of  $\mathcal{F}_{R,U}$ , we write  $\mathcal{E}_R$ ,  $\mathcal{F}_R$ , and  $\mathcal{G}_R$ , respectively. The vertices of  $\mathcal{E}_R$  are simply unimodular pairs from  $R \times R$  (so we write them as pairs rather than formal fractions). The vertices of  $\mathcal{G}_R$  are elements of the projective line over  $R$ . When  $R$  is a field,  $\mathcal{G}_R$  is the complete directed graph on the projective line.

The most familiar Farey complex is  $\mathcal{F}_{\mathbb{Z}}$ , the Farey complex of the integers (with units  $\{\pm 1\}$ ), shown in Figure 1.3. As usual, this complex is illustrated in the upper half-plane, with edges represented by hyperbolic lines. They are shown as undirected edges because  $-1 \in U$ .

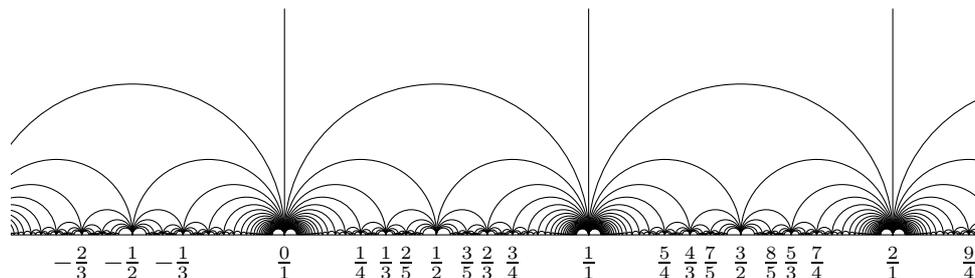


Figure 1.3. Part of the Farey complex  $\mathcal{F}_{\mathbb{Z}}$

Another familiar Farey complex is  $\mathcal{G}_{\mathbb{Z}[i]}$ , where  $\mathbb{Z}[i]$  is the ring of Gaussian integers. The full collection of units in  $\mathbb{Z}[i]$  is  $U = \{\pm 1, \pm i\}$ , so the vertices of  $\mathcal{G}_{\mathbb{Z}[i]}$  are Gaussian rationals. This complex gives rise to a tessellation of hyperbolic 3-space by ideal octahedra. A related Farey complex is  $\mathcal{G}_{\mathbb{Z}[\omega]}$ , where  $\omega = e^{\pi i/3}$ . Here  $U$  is the cyclic group generated by  $\omega$ , and  $\mathcal{G}_{\mathbb{Z}[\omega]}$  gives a

tessellation of hyperbolic 3-space by ideal tetrahedra. This Farey complex was considered in the context of  $SL_2$ -tilings by Felikson et al. in [12]. More generally, when  $R$  is the ring of integers of the imaginary quadratic field  $\mathbb{Q}(\sqrt{-d})$  with  $d = 1, 2, 3, 7, 11$ , the Farey complex  $\mathcal{G}_R$  gives rise to a tessellation of hyperbolic 3-space by ideal polyhedra associated with Bianchi groups. These tessellations were described by Hatcher [13] (among others); see [23] for a more general approach.

Much of this work concerns Farey complexes over finite rings, with focus on the ring of integers modulo  $N$ , denoted by  $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \dots, N - 1\}$ . The Farey complexes  $\mathcal{F}_{\mathbb{Z}/N\mathbb{Z}}$  (or, more briefly,  $\mathcal{F}_N$ ), for  $N = 2, 3, 4, 5, 6$ , are illustrated in Figure 1.4. These are, in order, a triangle, tetrahedron, octahedron, icosahedron, and a triangulation of the hexagonal torus (opposite sides are identified). These complexes were studied by Ivriissimtzis, Singerman, and Strudwick in [16, 22] (and other works) as quotients of  $\mathcal{F}_{\mathbb{Z}}$ ; we return to this later.

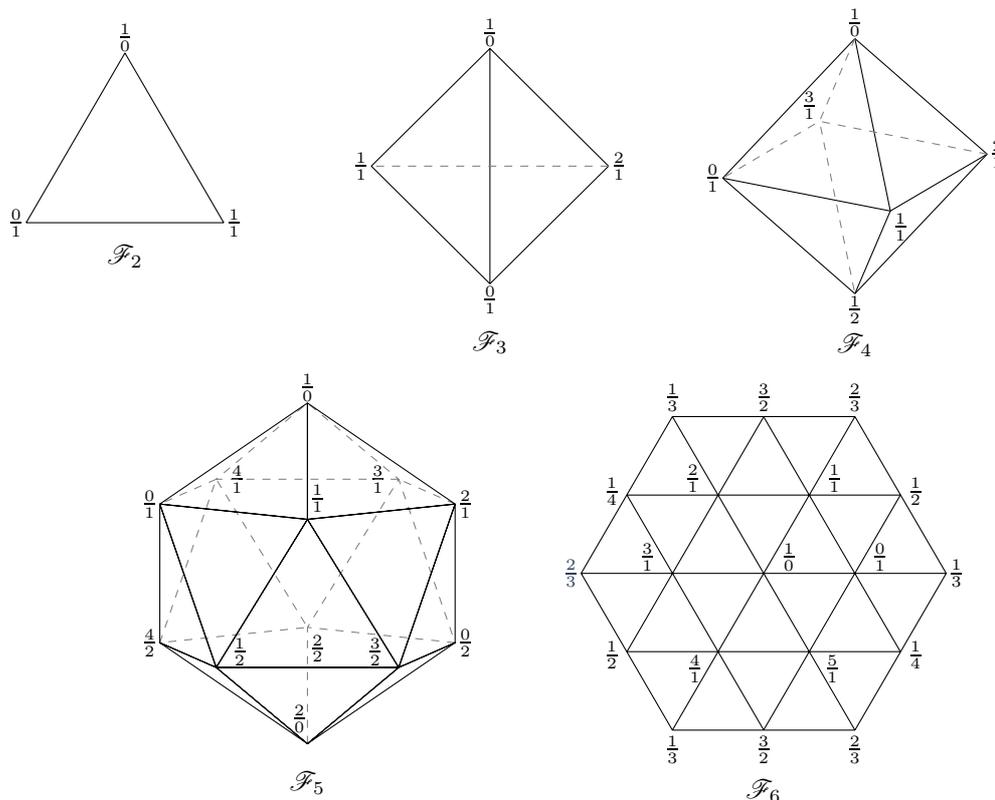


Figure 1.4. Farey complexes  $\mathcal{F}_2$ ,  $\mathcal{F}_3$ ,  $\mathcal{F}_4$ ,  $\mathcal{F}_5$ , and  $\mathcal{F}_6$

Two further Farey complexes are illustrated in Figure 1.5. Shown in Figure 1.5(a) is the Farey complex  $\mathcal{F}_R$ , where  $R$  is the field of size 4, which can be represented conveniently on a projective icosidodecahedron (opposite vertices on the outside are identified). In Figure 1.5(b) is the Farey complex  $\mathcal{F}_{R,U}$  for  $R = \mathbb{Z}[i]/3\mathbb{Z}[i]$ , which is the field of size 9, with  $U = \{\pm 1, \pm i\}$ . For clarity, the edges from  $1/0$  to the other black vertices are omitted, as are the edges from  $(1+i)/0$  to the other white vertices. This complex comprises 30 octahedra, with 9 incident to any given vertex, 4 incident to any given edge, and 2 incident to any given triangular face. With this representation, the complex (excluding its vertices) can be realised as a 3-manifold; in fact, it can be shown that this 3-manifold is  $SL_2(\mathbb{Z}[i], 3\mathbb{Z}[i]) \backslash \mathbb{H}^3$ , where  $SL_2(\mathbb{Z}[i], 3\mathbb{Z}[i])$  is the

principal congruence subgroup of  $\mathrm{SL}_2(\mathbb{Z}[i])$  for the ideal  $3\mathbb{Z}[i]$ . We refer the reader to [2] for more on these connections to hyperbolic 3-manifolds and for a proof (see [2, Theorem 1.1]) that  $\mathrm{SL}_2(\mathbb{Z}[i], 3\mathbb{Z}[i]) \backslash \mathbb{H}^3$  is topologically a 20-component link complement in the three-sphere.

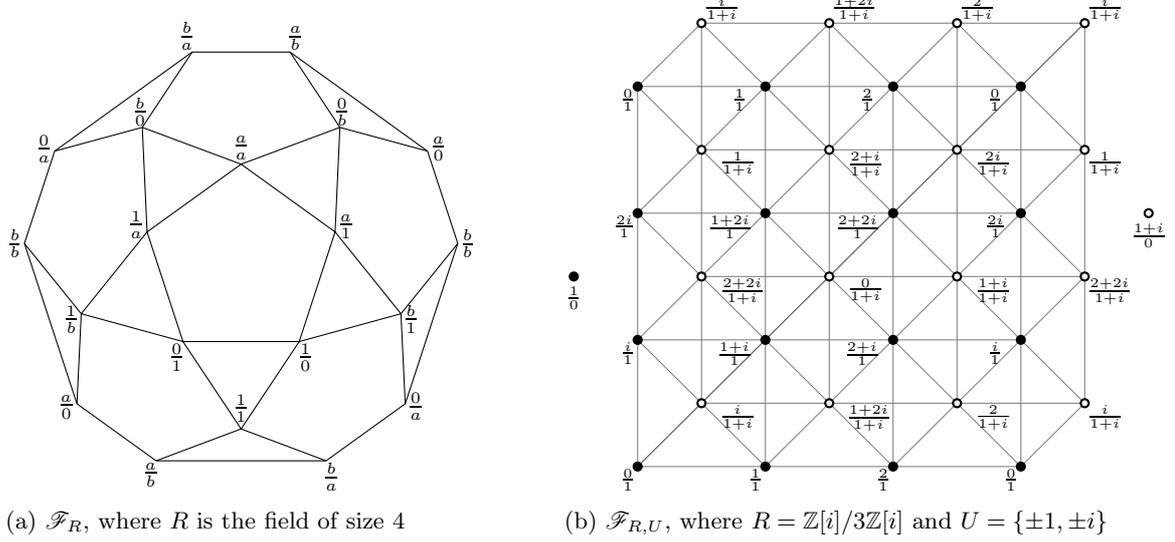


Figure 1.5. Farey complexes of the fields of size 4 and 9

For any Farey complex  $\mathcal{F}_{R,U}$ , there is a left action of  $\mathrm{SL}_2(R)$  on  $\mathcal{F}_{R,U}$  given by

$$\frac{x}{y} \mapsto \frac{ax + by}{cx + dy}, \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This action is transitive on directed edges. We denote the image of a vertex  $v$  (and matrix  $A$ ) under the action by  $Av$ .

The Farey complex  $\mathcal{F}_{R,U}$  is the quotient of  $\mathcal{E}_R$  under the action of  $U$ . Let  $\pi_U: \mathcal{E}_R \rightarrow \mathcal{F}_{R,U}$  be the associated quotient map, a covering map, given by  $(x, y) \mapsto U(x, y)$ . This map is equivariant under the action of  $\mathrm{SL}_2(R)$ , in the sense that, for any  $A \in \mathrm{SL}_2(R)$ , the following diagram commutes.

$$\begin{array}{ccc} \mathcal{E}_R & \xrightarrow{A} & \mathcal{E}_R \\ \pi_U \downarrow & & \downarrow \pi_U \\ \mathcal{F}_{R,U} & \xrightarrow{A} & \mathcal{F}_{R,U} \end{array}$$

A *bi-infinite path* in  $\mathcal{F}_{R,U}$  is a bi-infinite sequence of vertices  $\dots, v_{-1}, v_0, v_1, v_2, \dots$  such that  $v_{i-1} \rightarrow v_i$  is a directed edge in  $\mathcal{F}_{R,U}$ , for  $i \in \mathbb{Z}$ . We denote this path by  $\langle \dots, v_{-1}, v_0, v_1, v_2, \dots \rangle$ . We will also use finite paths, usually referred to simply as ‘paths’, with corresponding notation. The *length* of the finite path  $\langle v_0, v_1, \dots, v_n \rangle$  is  $n$ . Given any path  $\gamma$  in  $\mathcal{F}_{R,U}$  – whether finite or bi-infinite – there is a *lift* of this path to  $\mathcal{E}_R$ ; that is, there is a path  $\tilde{\gamma}$  in  $\mathcal{E}_R$  with  $\pi_U(\tilde{\gamma}) = \gamma$ , and usually there are many lifts (these are specified in Lemma 3.1).

We sketch proofs of these facts about Farey complexes and group actions in Sections 2 and 3.

Now, let us denote by  $\delta(u, v)$  the length of the shortest path from a vertex  $u$  to another vertex  $v$  in  $\mathcal{F}_{R,U}$ . We will prove that  $\mathcal{F}_{R,U}$  is connected if  $R$  is finite, so  $\delta(u, v)$  is defined for all pairs  $u$  and  $v$ . In fact, we prove the following stronger result about the *diameter* of a Farey complex, which is the maximum of  $\delta(u, v)$  among all pairs  $u$  and  $v$  in  $\mathcal{F}_{R,U}$ .

**Theorem 1.1.** *The diameter of the Farey complex  $\mathcal{E}_R$  of a finite ring  $R$  satisfies*

$$\text{diam } \mathcal{E}_R = \begin{cases} 1 & \text{if } R \text{ is } \mathbb{Z}/2\mathbb{Z}, \\ 2 & \text{if } R \text{ is } \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \text{ or } (\mathbb{Z}/2\mathbb{Z})^n, \text{ for } n > 1, \\ 3 & \text{otherwise.} \end{cases}$$

Here  $(\mathbb{Z}/2\mathbb{Z})^n$  is the direct product of  $n$  copies of  $\mathbb{Z}/2\mathbb{Z}$ . Since the map  $\mathcal{E}_R \rightarrow \mathcal{F}_{R,U}$  is a covering map, it follows that  $\text{diam } \mathcal{F}_{R,U} \leq \text{diam } \mathcal{E}_R \leq 3$ ; that is, the diameter of *any* Farey complex over a finite ring is at most 3. This generalises [22, Theorem 12] in which the same result was established for the Farey complexes  $\mathcal{F}_N$  (of  $\mathbb{Z}/N\mathbb{Z}$  with units  $\{\pm 1\}$ ). We prove Theorem 1.1 in Section 4, where we will also specify the diameter of  $\mathcal{F}_R$  for any finite ring  $R$ .

We saw in Figure 1.4 that the Farey complexes  $\mathcal{F}_N$  are surface complexes for low values of  $N$ , and in fact it is known that  $\mathcal{F}_N$  is a surface complex for all values of  $N$  – see [16] and Section 5. In that section we will prove that these are the *only* Farey complexes of finite rings that are surface complexes.

**Theorem 1.2.** *The Farey complex  $\mathcal{F}_{R,U}$  of a finite ring  $R$  with units  $U$ , where  $-1 \in U$ , is a surface complex if and only if  $R = \mathbb{Z}/N\mathbb{Z}$  and  $U = \{\pm 1\}$ .*

The chief objective of this paper is to use Farey complexes to provide combinatorial models for tame  $\text{SL}_2$ -tilings and friezes (over any ring), and the next three theorems, which are the principal results of this paper, achieve this objective, starting with  $\text{SL}_2$ -tilings.

The method we employ is motivated by that of [21] for  $\text{SL}_2$ -tilings over the integers. Let  $\mathcal{P}$  denote the collection of bi-infinite paths in  $\mathcal{E}_R$ , and let  $\mathbf{SL}_2$  denote the collection of tame  $\text{SL}_2$ -tilings over  $R$ . We define a function  $\tilde{\Phi}: \mathcal{P} \times \mathcal{P} \rightarrow \mathbf{SL}_2$  as follows. Given bi-infinite paths  $\gamma$  and  $\delta$  in  $\mathcal{P}$ , with vertices  $(a_i, b_i)$  and  $(c_j, d_j)$ , respectively, we define  $\tilde{\Phi}(\gamma, \delta)$  to be the  $\text{SL}_2$ -tiling  $\mathbf{M}$  with entries

$$m_{i,j} = a_i d_j - b_i c_j, \quad \text{for } i, j \in \mathbb{Z}.$$

We will prove that  $\mathbf{M}$  is a tame  $\text{SL}_2$ -tiling in Section 6, where we will also see that  $\tilde{\Phi}(A\gamma, A\delta) = \tilde{\Phi}(\gamma, \delta)$ , for  $A \in \text{SL}_2(R)$ .

Consider now some subgroup  $U$  of  $R^\times$ . Let  $\mathcal{P}_U$  denote the collection of bi-infinite paths in  $\mathcal{F}_{R,U}$  (so  $\mathcal{P} = \mathcal{P}_{\{1\}}$ ). The group  $\text{SL}_2(R)$  acts on  $\mathcal{P}_U \times \mathcal{P}_U$  by the rule  $(\gamma, \delta) \mapsto (A\gamma, A\delta)$ , where  $A \in \text{SL}_2(R)$ . There is also an action of the group  $U \times U$  on  $\mathbf{SL}_2$  given by

$$m_{i,j} \mapsto \lambda^{(-1)^i} \mu^{(-1)^j} m_{i,j},$$

where  $(\lambda, \mu) \in U \times U$ . Under this action, informally speaking, we multiply even columns of  $\mathbf{M}$  by  $\lambda$  and odd columns by  $\lambda^{-1}$ , and we multiply even rows of  $\mathbf{M}$  by  $\mu$  and odd rows by  $\mu^{-1}$ . This action is not usually faithful; it has kernel  $\{(\lambda, \lambda) : \lambda^2 = 1\}$ .



Consider now some subgroup  $U$  of  $R^\times$ , and let  $n$  be a positive integer, at least 2. We say that two vertices  $u$  and  $v$  of  $\mathcal{F}_{R,U}$  are *equivalent* if there exists  $\lambda \in R^\times$  such that  $v = \lambda u$ . We define  $\mathcal{C}_{n,U}$  to be the set of paths of length  $n$  in  $\mathcal{F}_{R,U}$  with initial and final vertices that are equivalent. We write  $\mathcal{C}_n$  for  $\mathcal{C}_{n,\{1\}}$ . There is an embedding of  $\mathcal{C}_{n,U}$  into  $\mathcal{P}_U$  in which the path  $\langle v_0, v_1, \dots, v_n \rangle$  (where  $v_n = \lambda v_0$ , for some  $\lambda \in R^\times$ ) is identified with the bi-infinite path with vertices  $v_i$  satisfying  $v_{i+n} = \lambda^{(-1)^i} v_i$ , for  $i \in \mathbb{Z}$ . To reduce notation, we write  $\mathcal{C}_{n,U}$  for the image under this embedding; thus, we may, for example, specify an element of  $\mathcal{C}_{n,U}$  by a finite path but consider it to be a bi-infinite path.

Let  $\mathbf{FR}_n$  denote the subset of  $\mathbf{SL}_2$  of tame friezes of width  $n$ , and let  $\mathbf{FR}_n^*$  denote the subset of tame semiregular friezes (second row 1's) of width  $n$ , where we identify a tame frieze with its extension to a tame  $\mathbf{SL}_2$ -tiling. Consider the map from  $\mathbf{FR}_n$  onto  $\mathbf{FR}_n^*$  given by  $m_{i,j} \mapsto \alpha^{(-1)^i} m_{i,j}$ , where  $\alpha = m_{1,0}$  (multiply even rows by  $\alpha$  and odd rows by  $\alpha^{-1}$ ). This induces a one-to-one correspondence  $R^\times \backslash \mathbf{FR}_n \rightarrow \mathbf{FR}_n^*$ . In this manner, every tame frieze of width  $n$  can be normalised to give a tame semiregular frieze of width  $n$ .

We will see in Section 8 that by restricting the domain of definition of  $\tilde{\Psi}$  to  $\mathcal{C}_n$  we obtain a map  $\tilde{\Psi}: \mathcal{C}_n \rightarrow \mathbf{FR}_n^*$ . We will also see that the  $\tilde{\Psi}$ -images of any two lifts to  $\mathcal{C}_n$  of two  $\mathbf{SL}_2(R)$ -equivalent paths in  $\mathcal{C}_{n,U}$  are  $U$ -equivalent in  $\mathbf{FR}_n^*$  under the action

$$m_{i,j} \mapsto \lambda^{(-1)^i + (-1)^j} m_{i,j},$$

where  $\lambda \in U$  (this map preserves the second row of 1's). As a consequence, we obtain an induced map  $\Psi_U: \mathbf{SL}_2(R) \backslash \mathcal{C}_{n,U} \rightarrow U \backslash \mathbf{FR}_n^*$  such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}_n & \xrightarrow{\tilde{\Psi}} & \mathbf{FR}_n^* \\ \downarrow & & \downarrow \\ \mathbf{SL}_2(R) \backslash \mathcal{C}_{n,U} & \xrightarrow{\Psi_U} & U \backslash \mathbf{FR}_n^* \end{array}$$

Our second main result is that  $\Psi_U$  is a bijection.

**Theorem 1.4.** *The map  $\Psi_U$  is a one-to-one correspondence between*

$$\mathbf{SL}_2(R) \backslash \left\{ \begin{array}{l} \text{paths of length } n \text{ between} \\ \text{equivalent vertices in } \mathcal{F}_{R,U} \end{array} \right\} \longleftrightarrow R^\times \backslash \left\{ \begin{array}{l} \text{tame semiregular friezes} \\ \text{over } R \text{ of width } n \end{array} \right\}.$$

This result generalises [21, Theorem 1.6], which is the special case  $R = \mathbb{Z}$  and  $U = \{\pm 1\}$ .

To illustrate Theorem 1.4, consider the tame semiregular frieze over  $\mathbb{Z}/5\mathbb{Z}$  shown in Figure 1.7.

The additional entry 4 above the frieze is a single entry from the extension of the frieze to an  $\mathbf{SL}_2$ -tiling. Consider now the slanted box. By placing the upper row over the lower row from this box we obtain the path

$$\frac{2}{0} \rightarrow \frac{2}{3} \rightarrow \frac{1}{2} \rightarrow \frac{0}{1} \rightarrow \frac{4}{0}$$

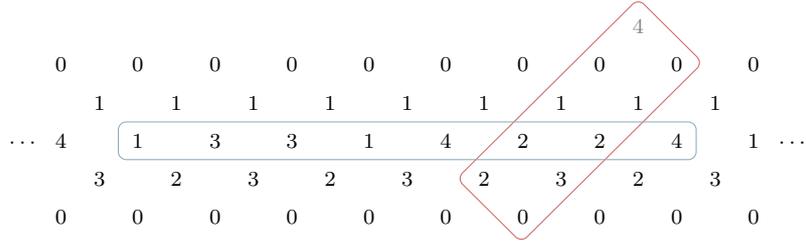


Figure 1.7. A tame semiregular frieze over  $\mathbb{Z}/5\mathbb{Z}$

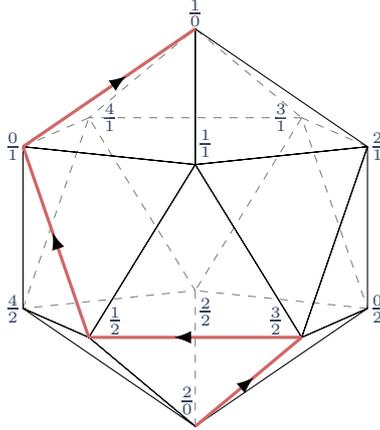


Figure 1.8. A path between equivalent vertices on  $\mathcal{F}_5$

in  $\mathcal{F}_5$ , which is illustrated in Figure 1.8. Note that  $4/0$  and  $1/0$  are equal in  $\mathcal{F}_5$ , as are  $2/3$  and  $3/2$ .

The path  $\gamma$  of Figure 1.8 corresponds to the frieze  $\mathbf{F}$  of Figure 1.7 under  $\Psi_U$  (with  $U = \{\pm 1\}$ ). The path has length 4, and accordingly  $\mathbf{F}$  has width 4. Considered as a bi-infinite path  $\gamma$  satisfies  $v_{i+4} = 2^{(-1)^i} v_i$ , for  $i \in \mathbb{Z}$ , so  $v_{i+8} = -v_i = v_i$ . Therefore  $\gamma$  has period 8 and, correspondingly, the frieze  $\mathbf{F}$  also has period 8 (indicated by the horizontal box in Figure 1.7).

We can obtain a similar result to Theorem 1.4 for tame *regular* friezes over  $R$ ; however, for these, single paths in  $\mathcal{F}_{R,U}$  cannot alone be used to distinguish regular and non-regular friezes, unless  $U = \{1\}$ . Consequently, we state our third main result using the Farey complex  $\mathcal{E}_R$ . We define a *semiclosed* path in  $\mathcal{E}_R$  to be a path with initial vertex  $v$  and final vertex  $-v$ , for some vertex  $v$  in  $\mathcal{E}_R$ . The semiclosed paths form a subcollection of  $\mathcal{C}_n$ , and by restricting  $\Psi$  (where  $\Psi = \Psi_{\{1\}}$ ) to this subcollection we obtain the following theorem, proven in Section 9.

**Theorem 1.5.** *The map  $\Psi$  is a one-to-one correspondence between*

$$\mathrm{SL}_2(R) \setminus \left\{ \begin{array}{l} \text{semiclosed paths of} \\ \text{length } n \text{ in } \mathcal{E}_R \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{tame regular friezes} \\ \text{over } R \text{ of width } n \end{array} \right\}.$$

The strength of Theorems 1.4 and 1.5 is that they allow us to represent tame friezes by paths, for which we can take advantage of the algebraic, combinatorial, and geometric properties of Farey complexes. We give three applications; the first is about quiddity cycles, introduced by Coxeter in [8].

Consider any tame semiregular frieze  $\mathbf{F}$  over a ring  $R$ . As noted earlier,  $\mathbf{F}$  is periodic. A *quiddity sequence* for  $\mathbf{F}$  is a period  $a_1, a_2, \dots, a_k$  from the third row  $m_{i+1, i-1}$ , for  $i \in \mathbb{Z}$ , of  $\mathbf{F}$ . For example, a quiddity sequence for the frieze of Figure 1.7 is indicated by the horizontal box. Quiddity sequences are never unique, for  $a_1, a_2, \dots, a_k, a_1, a_2, \dots, a_k$  is also a quiddity sequence for  $\mathbf{F}$  and so is  $a_2, a_3, \dots, a_k, a_1$ . We say that a *quiddity cycle* for  $\mathbf{F}$  is the collection of all cyclic permutations of a quiddity sequence  $a_1, a_2, \dots, a_k$ . Quiddity cycles are important in the theory of friezes because they encapsulate most of the information necessary to specify a tame semiregular frieze; we explore this later on.

Conway and Coxeter's celebrated observation [7] was that the quiddity cycles of tame friezes with positive integer entries are in correspondence with the collection of triangulated polygons. Here we prove that, for a finite ring, *any* finite sequence is a quiddity sequence of some frieze.

**Theorem 1.6.** *Any finite sequence in a finite ring  $R$  is a quiddity sequence for some tame semiregular frieze over  $R$ .*

Theorem 1.6 could be established from related results in the literature, such as [18, Theorem 1.15]. The merit of our approach is that the correspondence with paths facilitates a short, intuitive proof, in Section 10.

For our second application of Theorems 1.4 and 1.5, we enumerate tame friezes of a given width over a finite field. Theorem 1.4 reduces this task to that of enumerating closed paths in a complete graph, which is straightforward.

**Theorem 1.7.** *The number of tame friezes of width  $n$  over a finite field of size  $q$  is*

$$\frac{(q-1)(q^{n-1} + (-1)^n)}{q+1}.$$

Using Theorems 1.4 and 1.7 we also reprove a result [19, Theorem 1] of Morier-Genoud on the number of tame *regular* friezes of a given width over a finite field; see Section 11.

The third and most significant application of Theorems 1.4 and 1.5 concerns lifting tame  $\mathrm{SL}_2$ -tilings and friezes from  $\mathbb{Z}/N\mathbb{Z}$  to  $\mathbb{Z}$ . We will see in Section 12 that *any* tame  $\mathrm{SL}_2$ -tiling over  $\mathbb{Z}/N\mathbb{Z}$  lifts to a tame  $\mathrm{SL}_2$ -tiling over  $\mathbb{Z}$ . In fact, we will see that it is possible to choose a lift with all entries positive.

Lifting friezes is more complex; it is not always possible to lift a tame frieze over  $\mathbb{Z}/N\mathbb{Z}$  to a tame frieze over  $\mathbb{Z}$  of the same width. For example, the tame frieze over  $\mathbb{Z}/5\mathbb{Z}$  shown in Figure 1.7 cannot be lifted to a tame frieze over  $\mathbb{Z}$ , since the entries 2 and 3 in the second-last row are not congruent to  $\pm 1$  modulo 5. For another example, consider the frieze  $\mathbf{F}$  over  $\mathbb{Z}/6\mathbb{Z}$  shown in Figure 1.9. Alongside the frieze is a corresponding path  $\gamma$  in  $\mathcal{F}_6$  (specified by the correspondence  $\Psi_U$  of Theorem 1.4). This path winds once round the handle of the torus that underlies the surface complex of  $\mathcal{F}_6$ . Consequently, any lift  $\tilde{\gamma}$  of  $\gamma$  to  $\mathcal{F}_{\mathbb{Z}}$  is not closed, in which case  $\tilde{\gamma}$  does not specify a frieze, by Theorem 1.4, and hence  $\mathbf{F}$  cannot be lifted to a tame frieze over  $\mathbb{Z}$ . This reasoning will be made precise in Section 12.

To classify which tame friezes over  $\mathbb{Z}/N\mathbb{Z}$  lift to friezes over  $\mathbb{Z}$ , we introduce the following topological concepts. We say that a closed path in a Farey complex  $\mathcal{F}_R$  is *strongly contractible* if it can be transformed to a point by applying a finite number of the following two elementary homotopies. The first elementary homotopy is the removal of a single spur; that is, we replace a subpath  $\langle v, u, v \rangle$  of a closed path  $\gamma$  with the subpath  $\langle v \rangle$ , thereby decreasing the length of  $\gamma$  by 2. The second elementary homotopy is an elementary deformation over a triangular face of  $\mathcal{F}_R$ ;

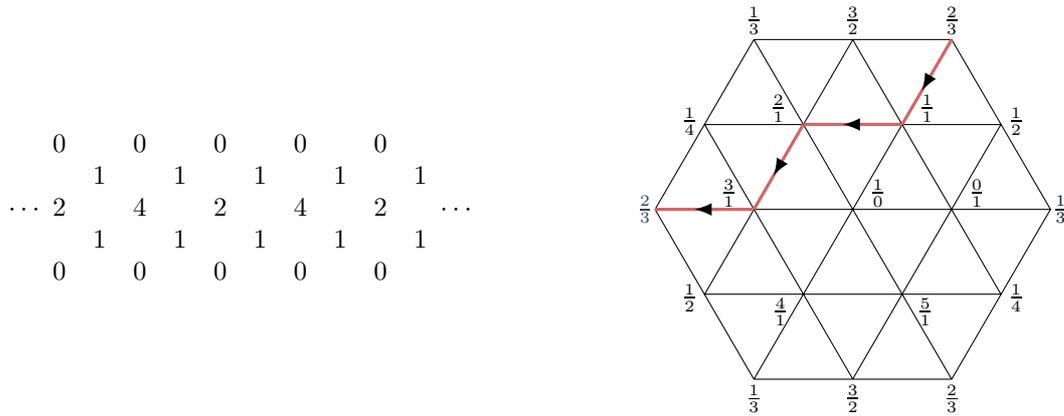


Figure 1.9. Tame regular frieze over  $\mathbb{Z}/6\mathbb{Z}$  with corresponding closed path in  $\mathcal{F}_6$

that is, we replace a subpath  $\langle u, v, w \rangle$  of  $\gamma$  with  $\langle u, w \rangle$ , where  $u, v$ , and  $w$  are mutually adjacent vertices, thereby decreasing the length of  $\gamma$  by 1.

Notice that being strongly contractible is *not* the same as being null homotopic (in the usual sense) because for strongly contractible paths we are only allowed to remove edges; we cannot add spurs or triangles. For example, consider the closed paths  $\gamma$  and  $\delta$  shown in Figure 1.10; both are null homotopic, however,  $\gamma$  is strongly contractible but  $\delta$  is not.

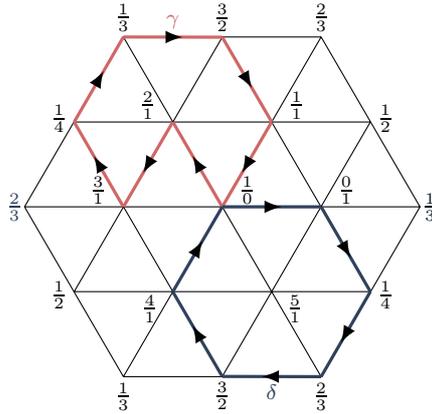


Figure 1.10. A strongly contractible closed path  $\gamma$  and a not strongly contractible closed path  $\delta$

Since  $\pm 1$  are the only units in  $\mathbb{Z}$ , any frieze  $\mathbf{F}$  over  $\mathbb{Z}/N\mathbb{Z}$  that lifts to a frieze  $\tilde{\mathbf{F}}$  over  $\mathbb{Z}$  must have entries 1 or  $-1$  in its second and second-last rows. Notice that  $\tilde{\mathbf{F}}$  is a lift of  $\mathbf{F}$  if and only if  $-\tilde{\mathbf{F}}$  is a lift of  $-\mathbf{F}$ , so we can restrict our attention to semiregular friezes. We say that  $\gamma$  is a path in  $\mathcal{F}_N$  corresponding to the tame semiregular frieze  $\mathbf{F}$  if the image under  $\Psi_{\{\pm 1\}}$  of the  $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ -orbit of  $\gamma$  is  $\mathbf{F}$ .

**Theorem 1.8.** *A tame semiregular frieze  $\mathbf{F}$  over  $\mathbb{Z}/N\mathbb{Z}$  lifts to a tame frieze over  $\mathbb{Z}$  of the same width if and only if any path  $\gamma$  in  $\mathcal{F}_N$  corresponding to  $\mathbf{F}$  is a strongly contractible closed path.*

A consequence of Theorem 1.8 is that all tame friezes over  $\mathbb{Z}/2\mathbb{Z}$  and  $\mathbb{Z}/3\mathbb{Z}$  lift to tame friezes

over  $\mathbb{Z}$ , because these rings are both fields (so equivalent vertices are equal) and the graphs of  $\mathcal{F}_2$  and  $\mathcal{F}_3$  are complete graphs, so all closed paths in these graphs are strongly contractible. However, the same cannot be said of  $\mathbb{Z}/4\mathbb{Z}$  since the closed path

$$\frac{0}{1} \rightarrow \frac{1}{1} \rightarrow \frac{2}{1} \rightarrow \frac{3}{1} \rightarrow \frac{0}{1}$$

is not strongly contractible.

## 2 Group actions on Farey complexes

In this section we discuss some of the properties of the action of  $\mathrm{SL}_2(R)$  on  $\mathcal{F}_{R,U}$  that were stated but not proven in the introduction. These properties are elementary, so we skip some details.

Let  $R$  be a ring and let  $U$  be a subgroup of  $R^\times$ . We consider the left group action  $\theta: \mathrm{SL}_2(R) \times \mathcal{F}_{R,U} \rightarrow \mathcal{F}_{R,U}$  given by the rule  $(A, x/y) \mapsto (ax + by)/(cx + dy)$ , where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R).$$

We write  $\theta_A(x/y)$  for  $\theta(A, x/y)$ . That this truly is a group action (for which  $\theta_A$  is an automorphism of  $\mathcal{F}_{R,U}$ ) will be established shortly. Remember that the formal fraction  $x/y$  represents the orbit  $U(x, y)$ . With this orbit notation, we have  $\theta_A(U(x, y)) = U(ax + by, cx + dy)$ .

The symbol  $\theta$  for the group action is used in this section alone, to facilitate the proofs. Outside this section we abandon  $\theta$  and simply write  $Av$  for  $\theta_A(v)$  and so forth.

**Proposition 2.1.** *The function  $\theta$  is a left group action of  $\mathrm{SL}_2(R)$  on  $\mathcal{F}_{R,U}$ .*

*Proof.* First we check that if  $(x, y)$  is a unimodular pair, then so is  $(x', y') = (ax + by, cx + dy)$ . We know that there exist  $r, s \in R$  with  $rx + sy = 1$ . We define  $r', s' \in R$  by  $\begin{pmatrix} r' & s' \end{pmatrix} = \begin{pmatrix} r & s \end{pmatrix} A^{-1}$ . Then

$$r'x' + s'y' = \begin{pmatrix} r' & s' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r & s \end{pmatrix} A^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} = 1,$$

so  $(x', y')$  is a unimodular pair.

Next we check that if  $x_1/y_1 = x_2/y_2$ , then  $(ax_1 + by_1)/(cx_1 + dy_1) = (ax_2 + by_2)/(cx_2 + dy_2)$ . This is indeed so, because if  $x_1/y_1 = x_2/y_2$  then there exists  $\lambda \in U$  with  $(x_1, y_1) = \lambda(x_2, y_2)$ , and the rest of the argument follows easily. We have now shown that  $\theta$  is a function. The left group action axioms can readily be seen to be satisfied.

It remains to show that  $\theta_A$  is a graph automorphism, for each  $A \in \mathrm{SL}_2(R)$ . For this we must show that  $\theta_A$  preserves incidence in  $\mathcal{F}_{R,U}$ . Suppose that there is a directed edge from  $x_1/y_1$  to  $x_2/y_2$ . Then

$$x_1y_2 - y_1x_2 = \begin{pmatrix} x_1 & y_1 \end{pmatrix} J \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in U, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let  $(x'_1, y'_1) = (ax_1 + by_1, cx_1 + dy_1)$  and  $(x'_2, y'_2) = (ax_2 + by_2, cx_2 + dy_2)$ . Then

$$x'_1y'_2 - y'_1x'_2 = \begin{pmatrix} x'_1 & y'_1 \end{pmatrix} J \begin{pmatrix} x'_2 \\ y'_2 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \end{pmatrix} A^T J A \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in U,$$

because  $A^T J A = J$  (where  $A^T$  is the transpose of  $A$ ). Hence there is a directed edge from  $x'_1/y'_1$  to  $x'_2/y'_2$ , as required.  $\square$

Typically, the action  $\theta$  is not faithful. Following the usual terminology for group actions, we consider the kernel of  $\theta$  to be the subgroup of  $\mathrm{SL}_2(R)$  of matrices  $A$  that satisfy  $\theta_A(v) = v$ , for all vertices  $v \in \mathcal{F}_{R,U}$ .

**Lemma 2.2.** *The kernel of  $\theta$  is*

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in U, \lambda^2 = 1 \right\}.$$

This can be proved by checking images of  $1/0$ ,  $0/1$ , and  $1/1$ ; we omit the details.

The action  $\theta$  is transitive on vertices, in the sense that, for any  $v, w \in \mathcal{F}_{R,U}$  there exists  $A \in \mathrm{SL}_2(R)$  with  $\theta_A(v) = w$ . In fact, if we consider  $\theta$  to act not on the vertices of  $\mathcal{F}_{R,U}$  but on directed edges (in the obvious way) then we can obtain the following stronger statement.

**Proposition 2.3.** *The action  $\theta$  is transitive on directed edges.*

*Proof.* Consider any directed edge from  $a/b$  to  $c/d$ . Let  $\lambda = ad - bc$ ; then  $\lambda \in U$ . We define  $a' = \lambda^{-1}a$  and  $b' = \lambda^{-1}b$ . Then

$$A = \begin{pmatrix} a' & c \\ b' & d \end{pmatrix} \in \mathrm{SL}_2(R),$$

and  $\theta_A(1/0) = a'/b' = a/b$  and  $\theta_A(0/1) = c/d$ . Consequently, any directed edge is the image of the directed edge from  $1/0$  to  $0/1$ , and the proposition then follows from the group action axioms.  $\square$

A consequence of Proposition 2.3 is that, when  $R$  is finite, the directed graph  $\mathcal{F}_{R,U}$  is regular, which means that all indegrees and outdegrees of vertices are equal. Since all the directed edges with source  $1/0$  have the form  $1/0 \rightarrow a/1$ , for  $a \in R$ , we see that the outdegree of each vertex is equal to the size of  $R$ , and the same can be said of the indegree. Notice that the indegree and outdegree are independent of  $U$ .

Consider now the map  $\pi_U: \mathcal{E}_R \rightarrow \mathcal{F}_{R,U}$  given by  $(x, y) \mapsto U(x, y)$  from the introduction. It is straightforward to check that this is a covering map. It was stated in the introduction that this map is equivariant under  $\theta$ ; we now prove this fact.

**Proposition 2.4.** *For any  $A \in \mathrm{SL}_2(R)$ , we have  $\pi_U \theta_A = \theta_A \pi_U$ .*

*Proof.* Let  $(x, y)$  be any vertex of  $\mathcal{E}_R$ . Then  $\theta_A(x, y) = U(ax + by, cx + d)$ , so

$$\pi_U \theta_A(x, y) = U(ax + by, cx + d) = \theta_A(U(x, y)) = \theta_A \pi_U(x, y),$$

as required.  $\square$

There was a sleight of hand in the statement of Proposition 2.4, in that the two occurrences of  $\theta_A$  differ: the left one acts on  $\mathcal{E}_R$  and the right one acts on  $\mathcal{F}_{R,U}$ .

### 3 Paths and itineraries

We gather here some basic properties of paths in Farey complexes. The first result we record is about lifting paths from the Farey complex  $\mathcal{F}_{R,U}$  of the ring  $R$  with units  $U$  to the Farey complex  $\mathcal{E}_R$  under the covering map  $\pi_U: \mathcal{E}_R \rightarrow \mathcal{F}_{R,U}$ .

**Lemma 3.1.** *Let  $\gamma$  be a bi-infinite path in  $\mathcal{F}_{R,U}$ . Then  $\gamma$  lifts to a bi-infinite path  $\tilde{\gamma}$  in  $\mathcal{E}_R$  with vertices  $(a_i, b_i)$ , and every other lift of  $\gamma$  to  $\mathcal{E}_R$  has the form  $\lambda^{(-1)^i}(a_i, b_i)$ , for  $\lambda \in U$ .*

*Proof.* Certainly  $\gamma$  has a lift to some path  $\tilde{\gamma}$  because  $\pi_U$  is a covering map. Let  $(a_i, b_i)$  be the vertices of  $\tilde{\gamma}$ . Suppose that another lift of  $\gamma$  has vertices  $(c_i, d_i)$ . Then  $(a_i, b_i)$  and  $(c_i, d_i)$  project to the same vertex in  $\mathcal{F}_{R,U}$ , so  $(c_i, d_i) = \lambda_i(a_i, b_i)$ , where  $\lambda_i \in U$ . Consequently,

$$\lambda_i \lambda_{i+1} = \lambda_i \lambda_{i+1} (a_i b_{i+1} - b_{i+1} a_i) = c_i d_{i+1} - d_i c_{i+1} = 1.$$

It follows that  $\lambda_i = \lambda_0^{(-1)^i}$ , for  $i \in \mathbb{Z}$ , as required.  $\square$

Next we explore the concept of itineraries of paths, which appeared in [21] for the Farey complex  $\mathcal{F}_{\mathbb{Z}}$ . Itineraries could be introduced through the theory of continued fractions, in the following sense. The convergents of a continued fraction determine a path in a Farey complex (see, for example, [4, 14]), and the coefficients of the continued fraction are the itinerary of this path. We opt not to take this approach here, for it would take us too far afield; nonetheless, those acquainted with the theory of continued fractions should find the material in this section to be familiar.

**Definition.** Let  $\gamma$  be a bi-infinite path with vertices  $(a_i, b_i)$  in the Farey complex  $\mathcal{E}_R$  of the ring  $R$ . The *itinerary* of  $\gamma$ , denoted  $\Sigma(\gamma)$ , is the bi-infinite sequence  $(e_i)$  in  $R$  given by

$$e_i = a_{i-1} b_{i+1} - b_{i-1} a_{i+1},$$

for  $i \in \mathbb{Z}$ .

The following alternative characterisation of itineraries will prove useful.

**Lemma 3.2.** *The itinerary of a bi-infinite path  $\gamma$  in  $\mathcal{E}_R$  with vertices  $(a_i, b_i)$  is the unique bi-infinite sequence  $(r_i)$  in  $R$  that satisfies*

$$a_{i-1} + a_{i+1} = r_i a_i \quad \text{and} \quad b_{i-1} + b_{i+1} = r_i b_i,$$

for  $i \in \mathbb{Z}$ .

*Proof.* Let  $(e_i)$  be the itinerary of  $\gamma$ . Then

$$a_{i-1} + a_{i+1} = a_{i-1}(a_i b_{i+1} - b_i a_{i+1}) + a_{i+1}(a_{i-1} b_i - b_{i-1} a_i) = e_i a_i,$$

and similarly  $b_{i-1} + b_{i+1} = e_i b_i$ . For uniqueness, let  $(r_i)$  be a bi-infinite sequence that satisfies  $a_{i-1} + a_{i+1} = r_i a_i$  and  $b_{i-1} + b_{i+1} = r_i b_i$ , for  $i \in \mathbb{Z}$ . Then  $(r_i - e_i)a_i = (r_i - e_i)b_i = 0$ , so

$$r_i - e_i = (r_i - e_i)(a_i b_{i+1} - b_i a_{i+1}) = 0.$$

Hence  $(r_i)$  and  $(e_i)$  coincide.  $\square$

We recall that  $\mathcal{P}$  denotes the collection of bi-infinite paths in  $\mathcal{E}_R$ . The next lemma shows that the itinerary map  $\Sigma: \mathcal{P} \rightarrow R^{\mathbb{Z}}$  is surjective.

**Lemma 3.3.** *For any bi-infinite sequence  $(r_i)$  in  $R$  and directed edge  $(a^*, b^*) \rightarrow (c^*, d^*)$  in  $\mathcal{E}_R$  there is a unique bi-infinite path  $(a_i, b_i)$  in  $\mathcal{E}_R$  with  $(a_0, b_0) = (a^*, b^*)$ ,  $(a_1, b_1) = (c^*, d^*)$ , and itinerary  $(r_i)$ .*

*Proof.* First we establish the existence of a path with the specified properties. We define  $(a_i, b_i)$ , for  $i \in \mathbb{Z}$ , from the equations

$$\begin{pmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{pmatrix} = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & r_i \end{pmatrix}, \quad \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}.$$

By taking determinants we see that  $(a_i, b_i)$  determine a path  $\gamma$ . Moreover, since  $a_{i-1} + a_{i+1} = r_i a_i$  and  $b_{i-1} + b_{i+1} = r_i b_i$  it follows from Lemma 3.2 that  $(r_i)$  is the itinerary of  $\gamma$ .

Next we show that this is the unique path with the specified properties. For this we can apply the recurrence relations  $a_{i-1} + a_{i+1} = r_i a_i$  and  $b_{i-1} + b_{i+1} = r_i b_i$  to deduce that  $(a_i, b_i)$  is determined uniquely from the itinerary  $(r_i)$  and directed edge  $(a^*, b^*) \rightarrow (c^*, d^*)$ .  $\square$

The itinerary map  $\Sigma$  is invariant under  $\mathrm{SL}_2(R)$ , in the sense that  $\Sigma(A\gamma) = \Sigma(\gamma)$ , for  $A \in \mathrm{SL}_2(R)$ . To see that this is so, notice that

$$e_i = \begin{pmatrix} a_{i-1} & b_{i-1} \end{pmatrix} J \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix}, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the itinerary  $(e'_i)$  of  $A\gamma$  satisfies

$$e'_i = \begin{pmatrix} a_{i-1} & b_{i-1} \end{pmatrix} A^T J A \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} = e_i,$$

because  $A^T J A = J$ , as required. It follows that  $\Sigma$  induces a map  $\mathrm{SL}_2(R) \backslash \mathcal{P} \rightarrow R^{\mathbb{Z}}$ . In fact, this map is bijective.

**Theorem 3.4.** *The map  $\Sigma$  induces a one-to-one correspondence between  $\mathrm{SL}_2(R) \backslash \mathcal{P}$  and  $R^{\mathbb{Z}}$ .*

*Proof.* We have only to show that the induced map is injective. Suppose then that  $\gamma$  and  $\delta$  are bi-infinite paths in  $\mathcal{P}$  with the same itinerary  $(r_i)$ . Let  $\gamma$  have vertices  $(a_i, b_i)$  and  $\delta$  have vertices  $(c_j, d_j)$ . By Proposition 2.3, there is a matrix  $A \in \mathrm{SL}_2(R)$  that maps the directed edge  $(a_0, b_0) \rightarrow (a_1, b_1)$  to the directed edge  $(c_0, d_0) \rightarrow (c_1, d_1)$ . Consequently,  $A\gamma$  and  $\delta$  have the same itinerary and the same 0th and 1st vertices. Hence they are equal, by Lemma 3.3, so the  $\mathrm{SL}_2(R)$ -orbits of  $\gamma$  and  $\delta$  are equal.  $\square$

We have explored itineraries on the Farey complex  $\mathcal{E}_R$ , and a similar account could be given for  $\mathcal{F}_R$ . For other Farey complexes  $\mathcal{F}_{R,U}$ , with a larger group of units  $U$ , the story is complicated by the presence of squares of units in the definition of an itinerary (which vanish if  $U = \{\pm 1\}$ ); there is no need for us to go into this.

We remark that itineraries can be defined for finite or half-infinite paths, and there are analogues of the results above in these restricted settings.

## 4 Graph distance on Farey complexes

Here we prove Theorem 1.1, which specifies the diameter of the Farey complex  $\mathcal{E}_R$  of a finite ring  $R$ . Later in the section we establish a similar theorem for  $\mathcal{F}_R$ .

To get started proving Theorem 1.1, we first classify Farey complexes  $\mathcal{E}_R$  of diameter 1. Before doing so, it is worth observing that if the characteristic of  $R$  (denoted  $\text{char } R$ ) is 2, then  $\mathcal{E}_R = \mathcal{F}_R$ .

**Lemma 4.1.** *The Farey complex  $\mathcal{E}_R$  of a finite ring  $R$  has diameter 1 if and only if  $R = \mathbb{Z}/2\mathbb{Z}$ .*

*Proof.* Certainly  $\mathcal{E}_{\mathbb{Z}/2\mathbb{Z}}$  has diameter 1 (see Figure 1.4). Suppose now that  $R \neq \mathbb{Z}/2\mathbb{Z}$ . Then  $R$  contains an element  $x$  other than 0 or 1, in which case there is no directed edge from  $(1, 0)$  to  $(1, x)$ , so  $\mathcal{E}_R$  does not have diameter 1, as required.  $\square$

A ring  $R$  is a *local ring* if it contains a unique maximal ideal  $M$ . We will use the well-known property of a local ring that if  $x, y \in R$  and  $x + y = 1$ , then one of  $x$  or  $y$  must be a unit. To see why this property holds, suppose that  $x$  and  $y$  are elements of a local ring  $R$  that are not units. Then  $Rx \neq R$  and  $Ry \neq R$ , and hence  $Rx$  and  $Ry$  are contained in the unique maximal ideal  $M$ . Consequently,  $Rx + Ry \neq R$ , so  $x + y \neq 1$ .

**Lemma 4.2.** *The only finite local rings with no units besides  $\pm 1$  are  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ , and  $\mathbb{Z}/4\mathbb{Z}$ .*

*Proof.* Suppose that  $R$  is a finite local ring with no units other than  $\pm 1$ . Suppose also that  $R \neq \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$ . Then there exists an element  $x$  of  $R$  other than 0 or  $\pm 1$ . Since  $R$  is a local ring,  $1 - x$  is a unit, in which case  $1 - x = -1$ , so  $x = 2$ . Hence  $R = \mathbb{Z}/4\mathbb{Z}$ .  $\square$

It is now straightforward to classify the finite local rings for which  $\mathcal{E}_R$  has diameter 2.

**Lemma 4.3.** *The Farey complex  $\mathcal{E}_R$  of a finite local ring  $R$  has diameter 2 if and only if  $R$  is  $\mathbb{Z}/3\mathbb{Z}$  or  $\mathbb{Z}/4\mathbb{Z}$ .*

*Proof.* To see that  $\mathcal{E}_{\mathbb{Z}/3\mathbb{Z}}$  has diameter 2, observe that there are paths

$$(1, 0) \rightarrow (0, 1) \rightarrow (2, 0), \quad (1, 0) \rightarrow (a, 1), \quad \text{and} \quad (1, 0) \rightarrow (2a + 2, 1) \rightarrow (a, 2)$$

of lengths 1 or 2 from  $(1, 0)$  to  $(2, 0)$ ,  $(a, 1)$ , and  $(a, 2)$ , respectively, for  $a \in \mathbb{Z}/3\mathbb{Z}$ . By applying suitable elements of  $\text{SL}_2(\mathbb{Z}/3\mathbb{Z})$  we can see that there is a path of length 1 or 2 from any vertex of  $\mathcal{E}_{\mathbb{Z}/3\mathbb{Z}}$  to another. Hence  $\mathcal{E}_{\mathbb{Z}/3\mathbb{Z}}$  has diameter 2. Reasoning similarly we can see that  $\mathcal{E}_{\mathbb{Z}/4\mathbb{Z}}$  has diameter 2.

Suppose now that  $R$  is a finite local ring other than  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ , or  $\mathbb{Z}/4\mathbb{Z}$ . By Lemma 4.2,  $R$  contains a unit  $x$  not equal to  $\pm 1$ . A quick calculation shows that there is no path of length 1 or 2 from  $(1, 0)$  to  $(x, 0)$ . Hence  $\text{diam } \mathcal{E}_R \geq 3$ . This confirms that  $\mathcal{E}_{\mathbb{Z}/3\mathbb{Z}}$  and  $\mathcal{E}_{\mathbb{Z}/4\mathbb{Z}}$  are the only Farey complexes of finite local rings with diameter 2.  $\square$

We can now prove Theorem 1.1.

*Proof of Theorem 1.1.* First we will prove that  $\text{diam } \mathcal{E}_R \leq 3$ .

Assume for the moment that  $R$  is a (finite) local ring. Let  $(a, b)$  be a vertex of  $\mathcal{E}_R$ ; then  $ax + by = 1$  for some elements  $x$  and  $y$  of  $R$ , so one of  $ax$  or  $by$  must be a unit. Hence one of  $a$  or  $b$  must be a unit. We will find a path from  $(1, 0)$  to  $(a, b)$  of length 3. Note that we include the possibility that  $(a, b) = (1, 0)$ .

Suppose first that  $a$  is a unit, with inverse  $q$ . Let  $r = -q(1 + b)$ . Then

$$(1, 0) \rightarrow (0, 1) \rightarrow (-1, r) \rightarrow (a, b)$$

is such a path. Suppose now that  $b$  is a unit, with inverse  $q$ . Let  $r = q(1 + a)$ . Then

$$(1, 0) \rightarrow (r + 1, 1) \rightarrow (r, 1) \rightarrow (a, b)$$

is a suitable path. Since  $\text{SL}_2(R)$  acts transitively on the vertices of  $\mathcal{E}_R$ , we see that there is a path of length 3 from any vertex to another.

Now let  $R$  be any finite ring; then  $R$  is a direct product of local rings (see, for example, [5, Theorem 3.1.4]). By taking products of paths, we can obtain a path of length 3 from any vertex to another. Consequently,  $\text{diam } \mathcal{E}_R \leq 3$ .

It remains to identify those Farey complexes  $\mathcal{E}_R$  with diameters 1 and 2. By Lemma 4.1, the only finite ring  $R$  for which  $\text{diam } \mathcal{E}_R = 1$  is  $R = \mathbb{Z}/2\mathbb{Z}$ .

Suppose now that  $R$  is either equal to  $\mathbb{Z}/3\mathbb{Z}$  or  $\mathbb{Z}/4\mathbb{Z}$ , or else it is equal to  $(\mathbb{Z}/2\mathbb{Z})^n$ , for  $n > 1$ . In the former case  $\text{diam } \mathcal{E}_R = 2$ , by Lemma 4.2. In the latter case, observe that we can find a path of length exactly 2 from any vertex  $u$  of  $\mathcal{E}_{\mathbb{Z}/2\mathbb{Z}}$  to any other vertex  $v$ , even when  $v = u$ . Hence we can find a path of length 2 from any vertex of  $\mathcal{E}_R$  to another, so  $\text{diam } \mathcal{E}_R = 2$ .

Finally, consider any finite ring  $R$  for which  $\text{diam } \mathcal{E}_R = 2$ . We can express  $R$  as a product of finite local rings. The diameter of the Farey complex of each of these local rings must be at most 2. Hence, by Lemmas 4.1 and 4.2, each local ring must be one of  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ , or  $\mathbb{Z}/4\mathbb{Z}$ . However, in  $\mathcal{E}_{\mathbb{Z}/3\mathbb{Z}}$  and  $\mathcal{E}_{\mathbb{Z}/4\mathbb{Z}}$  there is a path of length 3 from any vertex to itself but not a path of length 2, as one can easily verify. Consequently, if there is more than one local ring in the product  $R$ , and if one of those local rings is  $\mathbb{Z}/3\mathbb{Z}$  or  $\mathbb{Z}/4\mathbb{Z}$ , then we can find two distinct vertices in  $\mathcal{E}_R$  not connected by a path of length 1 or 2. (For example, if  $u \in \mathcal{E}_{\mathbb{Z}/3\mathbb{Z}}$  and  $v$  and  $w$  are distinct vertices in a ring  $S$ , then there is no path of length 1 or 2 from  $(u, v)$  to  $(u, w)$  in  $\mathcal{E}_R$ , where  $R = \mathbb{Z}/3\mathbb{Z} \times S$ .) In this case, then,  $\text{diam } \mathcal{E}_R = 3$ .

This argument shows that the only finite rings  $R$  for which  $\text{diam } \mathcal{E}_R = 2$  are  $\mathbb{Z}/3\mathbb{Z}$ ,  $\mathbb{Z}/4\mathbb{Z}$ , and  $(\mathbb{Z}/2\mathbb{Z})^n$ , for  $n > 1$ .  $\square$

Next we state a similar result to Theorem 1.1 for the Farey complex  $\mathcal{F}_R$  instead of  $\mathcal{E}_R$ .

**Theorem 4.4.** *The diameter of the Farey complex  $\mathcal{F}_R$  of a finite ring  $R$  satisfies*

$$\text{diam } \mathcal{F}_R = \begin{cases} 1 & \text{if } R \text{ is } \mathbb{Z}/2\mathbb{Z} \text{ or } \mathbb{Z}/3\mathbb{Z}, \\ 2 & \text{if } R \text{ is a direct product of rings } \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \text{ and } \mathbb{Z}/4\mathbb{Z} \\ & \text{(and } R \neq \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}), \\ 3 & \text{otherwise.} \end{cases}$$

Here the direct product in the diameter 2 case can involve multiple copies of  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ , and  $\mathbb{Z}/4\mathbb{Z}$ ; for example, the diameter of  $\mathcal{F}_R$  is 2, where  $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ .

We prove Theorem 4.4 in a similar way to how we proved Theorem 1.1, beginning by classifying the Farey complexes  $\mathcal{F}_R$  of diameter 1.

**Lemma 4.5.** *The Farey complex  $\mathcal{F}_R$  of a finite ring  $R$  has diameter 1 if and only if  $R$  is  $\mathbb{Z}/2\mathbb{Z}$  or  $\mathbb{Z}/3\mathbb{Z}$ .*

*Proof.* Certainly  $\mathcal{F}_2$  and  $\mathcal{F}_3$  have diameter 1 (see Figure 1.4). Suppose now that  $R$  is not equal to  $\mathbb{Z}/2\mathbb{Z}$  or  $\mathbb{Z}/3\mathbb{Z}$ . Then  $R$  contains an element  $x$  other than 0 or  $\pm 1$ , in which case the vertices  $1/0$  and  $1/x$  of  $\mathcal{F}_R$  are not adjacent. Hence  $\mathcal{F}_R$  does not have diameter 1, as required.  $\square$

Next we classify the finite local rings for which  $\mathcal{F}_R$  has diameter 2.

**Lemma 4.6.** *The Farey complex  $\mathcal{F}_R$  of a finite local ring  $R$  has diameter 2 if and only if  $R$  is  $\mathbb{Z}/4\mathbb{Z}$ .*

*Proof.* Certainly  $\mathcal{F}_4$  has diameter 2 since it is an octahedron, as shown in Figure 1.4, and  $\mathcal{F}_2$  and  $\mathcal{F}_3$  have diameter 1.

Suppose now that  $R$  is a finite local ring other than  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ , or  $\mathbb{Z}/4\mathbb{Z}$ . By Lemma 4.2,  $R$  contains a unit  $x$  not equal to  $\pm 1$ . Then one can check that the graph distance between  $1/0$  and  $x/0$  is at least 3, so  $\text{diam } \mathcal{F}_3 \neq 2$ , as required.  $\square$

We can now prove Theorem 4.4.

*Proof of Theorem 4.4.* We know that  $\text{diam } \mathcal{E}_R \leq 3$ , by Theorem 1.1, and since  $\pi_{\{\pm 1\}}: \mathcal{E}_R \rightarrow \mathcal{F}_R$  is a covering map we see that  $\text{diam } \mathcal{F}_R \leq 3$ . By Lemma 4.5, among Farey complexes of finite rings only  $\mathcal{F}_2$  and  $\mathcal{F}_3$  have diameter 1.

It remains to determine the Farey complexes  $R$  for which  $\text{diam } \mathcal{F}_R = 2$ . To this end, suppose that  $R$  is some direct product of copies of  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ , and  $\mathbb{Z}/4\mathbb{Z}$  (and  $R \neq \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$ ). Between any two (possibly equal) vertices of any one of  $\mathcal{F}_2$ ,  $\mathcal{F}_3$ , and  $\mathcal{F}_4$  we can find a path of length exactly 2. Hence we can find a path of length 2 between any two vertices of  $\mathcal{F}_R$ , so  $\text{diam } \mathcal{F}_R = 2$ .

Finally, consider any finite ring  $R$  for which  $\text{diam } \mathcal{F}_R = 2$ . We can express  $R$  as a product of finite local rings. The diameter of the Farey complex of each of these local rings must be at most 2. Hence, by Lemmas 4.5 and 4.6, each local ring must be one of  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ , or  $\mathbb{Z}/4\mathbb{Z}$ , as required.  $\square$

## 5 Farey complexes and surfaces

In this section we prove Theorem 1.2, which says that the Farey complex  $\mathcal{F}_{R,U}$  of a finite ring  $R$  with units  $U$  is a surface complex if and only if  $R = \mathbb{Z}/N\mathbb{Z}$  and  $U = \{\pm 1\}$ . We begin by discussing the Farey complex  $\mathcal{F}_N$  for  $\mathbb{Z}/N\mathbb{Z}$ , which, as we shall see, can be realised as a triangulation of a hyperbolic surface with the vertices of  $\mathcal{F}_N$  located at the cusps of the surface and the edges of  $\mathcal{F}_N$  represented by hyperbolic geodesics between pairs of cusps. We summarise the details, since they appear elsewhere, notably in [16].

The group  $\text{SL}_2(\mathbb{Z})$  acts on the extended complex plane by the familiar rule for Möbius transformations

$$z \mapsto \frac{az + b}{cz + d}, \quad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

with the usual conventions for the point  $\infty$ . This group also acts on the extended rationals  $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$  and on the upper half-plane  $\mathbb{H} = \{z : \text{Im } z > 0\}$ . The action of  $\text{SL}_2(\mathbb{Z})$  on  $\mathbb{H}$  preserves the hyperbolic metric  $|dz|/\text{Im } z$  on  $\mathbb{H}$ . For  $N = 2, 3, \dots$ , the *principal congruence subgroup* of  $\text{SL}_2(\mathbb{Z})$  of level  $N$  is the group

$$\Gamma_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

It is a normal subgroup of  $\text{SL}_2(\mathbb{Z})$  because it is the kernel of the homomorphism

$$\text{SL}_2(\mathbb{Z}) \longrightarrow \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$$

induced by the reduction homomorphism  $\mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z}$  that sends an integer  $a$  to the class  $\bar{a}$  of integers congruent to  $a$  modulo  $n$ . The group  $\Gamma_N$  is discrete and acts freely on  $\mathbb{H}$ , so the quotient  $S_N = \Gamma_N \backslash \mathbb{H}$  is a hyperbolic surface. Topologically,  $S_N$  is a compact surface with finitely many punctures. Geometrically, each puncture is said to be a *cusp* of  $S_N$ ; there is a neighbourhood of each cusp isometric to the hyperbolic punctured unit disc. For  $N > 2$ , we have

$$\text{genus} = 1 + \frac{N^2(N-6)}{24} \prod_{p|N} \left(1 - \frac{1}{p^2}\right), \quad \text{number of cusps} = \frac{N^2}{2} \prod_{p|N} \left(1 - \frac{1}{p^2}\right),$$

where both products are taken over the prime divisors  $p$  of  $N$ . When  $N = 2$  the genus is 0 and there are 3 cusps; see [11, Section 3.9] for these formulae.

The surface  $S_N$  can be compactified by adjoining one additional point to  $S_N$  for each cusp. This can be realised by considering the action of  $\Gamma_N$  on  $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q}_\infty$ . Under this action, the cusps of  $S_N$  are identified with the orbits of  $\Gamma_N$  on  $\mathbb{Q}_\infty$ . We denote the resulting compactification of  $S_N$  by  $\overline{S_N} = \Gamma_N \backslash \overline{\mathbb{H}}$ ; full details of its construction can be found in [11, Chapter 2]. Under the projection  $\sigma: \overline{\mathbb{H}} \longrightarrow \overline{S_N}$  the usual Farey complex  $\mathcal{F}_\mathbb{Z}$  on  $\overline{\mathbb{H}}$  (see Figure 1.3) is mapped to a triangulation of  $\overline{S_N}$  with vertices at the cusps of  $S_N$  and edges represented by hyperbolic geodesics between certain pairs of cusps. We will show that this triangulation is a realisation of the Farey complex  $\mathcal{F}_N$  for  $N > 2$ . The reader can consult [16] for a more complete explanation.

We recall that vertices of  $\mathcal{F}_N$  have the form  $\pm(x, y)$ , where  $x, y \in \mathbb{Z}/N\mathbb{Z}$ . There is a one-to-one correspondence between the cusps  $\Gamma_N \backslash \mathbb{Q}_\infty$  and the vertices of  $\mathcal{F}_N$  given by

$$\Gamma_N(a/b) \longmapsto \pm(\bar{a}, \bar{b}).$$

Two cusps  $\Gamma_N(a/b)$  and  $\Gamma_N(c/d)$  are adjacent in  $\sigma(\mathcal{F}_\mathbb{Z})$  if and only if there exist  $A, B \in \Gamma_N$  with

$$\begin{pmatrix} a & b \end{pmatrix} A^T J B \begin{pmatrix} c \\ d \end{pmatrix} = \pm 1, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and this occurs if and only if  $\bar{a}\bar{d} - \bar{b}\bar{c} = \pm 1$ .

For  $N > 2$ , the image of every face of  $\mathcal{F}_\mathbb{Z}$  is uniquely specified by the image of its three vertices. The case  $N = 2$  is special:  $\mathcal{F}_2$  consists of a single face, while  $\overline{S_2}$  is topologically a sphere arising from two copies of that face glued along the boundary.

This completes our summary of how  $\mathcal{F}_N$  arises as a surface complex, indeed one endowed with a hyperbolic structure and with vertices located at the cusps of the surface. There is a similar story in three dimensions, exemplified by the octahedral tessellation of a hyperbolic 3-manifold in Figure 1.5(b).

We now turn to proving Theorem 1.2. For this (and the remainder of this section) we assume that  $R$  is a finite ring and  $U$  is a subgroup of  $R^\times$  that contains  $\{\pm 1\}$ .

We consider a 2-complex  $C$  to consist of a collection of vertices  $V$ , a collection of edges  $E$  comprising subsets of  $V$  of size 2, and a collection of faces  $F$  comprising subsets of  $V$  of size 3, where each subset of a face that has size 2 belongs to  $E$ . This definition does not allow multiple edges between a pair of vertices and nor does it allow edges with only one vertex. Familiar concepts like connectedness can be defined for 2-complexes in the usual way. The Farey complex  $\mathcal{F}_{R,U}$  is a finite 2-complex, because  $\pm 1 \in U$  (so directed edges come in inverse pairs, which can be considered as undirected edges). It is connected, by Theorem 1.1. Following [6, Chapter 3], we define a *surface complex*  $C$  to be a connected 2-complex that satisfies the following two properties (simplified slightly for our 2-complexes).

- (1) Each edge of  $C$  is incident to either one or two faces.
- (2) For any two edges  $u$  and  $v$  incident to the same vertex  $w$  in  $C$  there is a sequence of distinct neighbours  $v_1, v_2, \dots, v_n$  of  $w$  with  $v_1 = u$  and  $v_n = v$ , such that  $v_{i-1}$  and  $v_i$  are adjacent, for  $i = 2, 3, \dots, n$ .

A surface complex can be realised as a topological surface, possibly with boundary. An edge lies on the boundary if it is incident to only one face.

In light of Property (1) we examine the faces incident to an edge in  $\mathcal{F}_{R,U}$ .

**Lemma 5.1.** *The faces of  $\mathcal{F}_{R,U}$  incident to the edge between adjacent vertices  $a/b$  and  $c/d$  are the triples with third vertex*

$$\frac{a\mu + c}{b\mu + d}, \quad \text{for } \mu \in U,$$

and each such triple is distinct.

*Proof.* The neighbours of  $1/0$  have the form  $x/1$ , for  $x \in R$ ; of these, those vertices of the form  $\mu/1$ , for  $\mu \in U$ , are also neighbours of  $0/1$ . Consequently, the faces of  $\mathcal{F}_{R,U}$  incident to the edge between  $1/0$  and  $0/1$  are the triples with third vertex  $\mu/1$ , and each such triple is distinct.

Consider now two adjacent vertices  $a/b$  and  $c/d$ . Let  $\lambda = ad - bc$ . Then  $\lambda \in U$ , so  $a'd - b'c = 1$ , where  $a' = \lambda^{-1}a$  and  $b' = \lambda^{-1}b$ , and hence

$$\begin{pmatrix} a' & c \\ b' & d \end{pmatrix} \in \mathrm{SL}_2(R).$$

This matrix sends  $1/0$  to  $a/b$ ,  $0/1$  to  $c/d$ , and  $\mu/1$  to  $(a\mu' + c)/(b\mu' + d)$ , where  $\mu' = \lambda^{-1}\mu$ . Since  $\mathrm{SL}_2(R)$  acts by automorphisms of  $\mathcal{F}_{R,U}$ , we see that the faces incident to  $a/b$  and  $c/d$  have the required form.  $\square$

Lemma 5.1 shows that Property (1) fails for  $\mathcal{F}_{R,U}$  unless  $U$  is  $\{\pm 1\}$ . Next we consider Property (2), focusing on the complex  $\mathcal{F}_R$ .

**Lemma 5.2.** *If  $\mathcal{F}_R$  satisfies Property (2), then  $R = \mathbb{Z}/N\mathbb{Z}$ , for some  $N \geq 2$ .*

*Proof.* Suppose that  $\mathcal{F}_R$  satisfies Property (2). Suppose also that  $R \neq \mathbb{Z}/N\mathbb{Z}$ , so we can find  $a \in R$  distinct from  $0, 1, \dots, N-1$ , where  $N = \mathrm{char} R$ . Let  $w = 1/0$ . For each neighbour  $x/1$  of  $w$ , the only neighbours of both  $x/1$  and  $w$  are  $(x \pm 1)/1$ . It follows that  $u = 0/1$  and  $v = a/1$  fail Property (2), as required.  $\square$

We now have all the ingredients for proving Theorem 1.2. The Farey complex  $\mathcal{F}_{R,U}$  is certainly a surface complex when  $R = \mathbb{Z}/N\mathbb{Z}$  and  $U = \{\pm 1\}$ , as we saw earlier. On the other hand, if  $U$  is not  $\{\pm 1\}$ , then  $\mathcal{F}_{R,U}$  fails Property (1), by Lemma 5.1, so it is not a surface complex. And if  $U$  is  $\{\pm 1\}$ , then  $\mathcal{F}_{R,U}$  only satisfies Property (2) if  $\mathcal{F}_{R,U} = \mathcal{F}_N$ , for some  $N \geq 2$ , by Lemma 5.2. This completes the proof of Theorem 1.2.

## 6 Tame $\mathrm{SL}_2$ -tilings

In this section we prove Theorem 1.3. Before that we will establish the following theorem, which gives necessary and sufficient conditions for an  $\mathrm{SL}_2$ -tiling to be tame. This result is well-known for  $\mathrm{SL}_2$ -tilings over the integers; we are not aware of a proof of the more general statement.

**Theorem 6.1.** *An  $\mathrm{SL}_2$ -tiling  $\mathbf{M}$  over a ring  $R$  is tame if and only if there are bi-infinite sequences  $(r_i)$  and  $(s_j)$  in  $R$  such that*

$$m_{i-1,j} + m_{i+1,j} = r_i m_{i,j} \quad \text{and} \quad m_{i,j-1} + m_{i,j+1} = s_j m_{i,j},$$

for  $i, j \in \mathbb{Z}$ . Furthermore, if  $\mathbf{M}$  is tame, then  $(r_i)$  and  $(s_j)$  are uniquely determined by  $\mathbf{M}$ .

Key to proving Theorem 6.1 is the following elementary lemma (not proven here), which is a special case of a procedure known as ‘Dodgson condensation’ for calculating determinants.

**Lemma 6.2.** *In any ring  $R$  we have*

$$e \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \det \begin{pmatrix} a & b \\ d & e \end{pmatrix} \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - \det \begin{pmatrix} b & c \\ e & f \end{pmatrix} \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}.$$

We require another lemma before proving Theorem 6.1.

**Lemma 6.3.** *Consider any 3-by-3 matrix*

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

over a ring  $R$  with determinant 0, and suppose that

$$\det \begin{pmatrix} a & b \\ d & e \end{pmatrix} = \det \begin{pmatrix} b & c \\ e & f \end{pmatrix} = \det \begin{pmatrix} d & e \\ g & h \end{pmatrix} = \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} = 1.$$

Then  $d + f = \Delta e$ , where  $\Delta = af - cd = di - fg$ .

*Proof.* Let  $\Delta_1 = af - cd$  and  $\Delta_2 = di - fg$ . By moving the left column of  $A$  over to the right and applying Lemma 6.2 we obtain

$$f \det A = \Delta_1 - \Delta_2.$$

Since  $\det A = 0$ , we see that  $\Delta_1 = \Delta_2$ . Then

$$d + f = (bf - ce)d + (ae - bd)f = (af - cd)e = \Delta e,$$

as required. □

Now we can prove Theorem 6.1.

*Proof of Theorem 6.1.* Suppose first that  $m_{i+1,j} + m_{i-1,j} = r_i m_{i,j}$  and  $m_{i,j+1} + m_{i,j-1} = s_j m_{i,j}$ , for  $i, j \in \mathbb{Z}$ . Then the rows (or columns) of any 3-by-3 submatrix of  $\mathbf{M}$  are linearly dependent, so the determinant of that matrix is 0. Hence  $\mathbf{M}$  is tame.

Suppose now that  $\mathbf{M}$  is tame. Then, by Lemma 6.3, for any  $i, j \in \mathbb{Z}$  we have

$$m_{i,j-1} + m_{i,j+1} = \Delta_{i,j} m_{i,j},$$

where  $\Delta_{i,j} = m_{i-1,j-1} m_{i,j+1} - m_{i-1,j+1} m_{i,j-1} = m_{i,j-1} m_{i+1,j+1} - m_{i,j+1} m_{i+1,j-1}$ . This last pair of equations shows that, for any integer  $j$ , we have  $\Delta_{i,j} = \Delta_{k,j}$ , for all  $i, k \in \mathbb{Z}$ . Consequently, there is  $s_j \in R$  with

$$m_{i,j-1} + m_{i,j+1} = s_j m_{i,j}, \quad \text{for } i \in \mathbb{Z},$$

as required. A similar argument gives the existence of the required sequence  $(r_i)$ .

It remains to prove that  $(r_i)$  and  $(s_j)$  are uniquely determined by  $\mathbf{M}$ . Again, we prove this for  $(s_j)$ ; the proof for  $(r_i)$  is similar. Suppose then that there is another sequence  $(s'_j)$  in  $R$  with  $m_{i,j-1} + m_{i,j+1} = s'_j m_{i,j}$ , for  $i, j \in \mathbb{Z}$ . Choose any integer  $j$ . Then  $(s_j - s'_j) m_{i,j} = 0$ , for  $i \in \mathbb{Z}$ . Hence

$$s_j - s'_j = (s_j - s'_j)(m_{0,j} m_{1,j+1} - m_{0,j+1} m_{1,j}) = 0,$$

so  $s_j = s'_j$ , as required.  $\square$

Next we set about proving Theorem 1.3. Recall (from the introduction) that  $\mathcal{P}$  denotes the collection of bi-infinite paths in  $\mathcal{E}_R$  and  $\mathbf{SL}_2$  denotes the collection of tame  $\mathbf{SL}_2$ -tilings over  $R$ . The function  $\tilde{\Phi}: \mathcal{P} \times \mathcal{P} \rightarrow \mathbf{SL}_2$  sends the pair of paths  $\gamma$  and  $\delta$ , with vertices  $(a_i, b_i)$  and  $(c_j, d_j)$ , to the  $\mathbf{SL}_2$ -tiling  $\mathbf{M}$  with entries

$$m_{i,j} = \begin{pmatrix} a_i & b_i \end{pmatrix} J \begin{pmatrix} c_j \\ d_j \end{pmatrix} = a_i d_j - b_i c_j, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We can see that  $\mathbf{M}$  is indeed an  $\mathbf{SL}_2$ -tiling by observing that

$$\begin{pmatrix} m_{i,j} & m_{i,j+1} \\ m_{i+1,j} & m_{i+1,j+1} \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{pmatrix} J \begin{pmatrix} c_j & c_{j+1} \\ d_j & d_{j+1} \end{pmatrix} \in \mathbf{SL}_2(R). \quad (6.1)$$

We must also show that  $\mathbf{M}$  is tame. By Lemma 3.2, the itinerary  $(e_i)$  of  $\gamma$  satisfies  $a_{i-1} + a_{i+1} = e_i a_i$  and  $b_{i-1} + b_{i+1} = e_i b_i$ , for  $i \in \mathbb{Z}$ . It follows that any three consecutive rows of  $\mathbf{M}$  are linearly dependent, so the determinant of any 3-by-3 submatrix of  $\mathbf{M}$  is 0, as required.

**Lemma 6.4.** *The map  $\tilde{\Phi}$  satisfies  $\tilde{\Phi}(A\gamma, A\delta) = \tilde{\Phi}(\gamma, \delta)$ , for  $A \in \mathbf{SL}_2(R)$ .*

*Proof.* Let

$$\begin{pmatrix} a'_i \\ b'_i \end{pmatrix} = A \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \quad \begin{pmatrix} c'_j \\ d'_j \end{pmatrix} = A \begin{pmatrix} c_j \\ d_j \end{pmatrix}, \quad \text{and } m'_{i,j} = a'_i d'_j - b'_i c'_j,$$

for  $i, j \in \mathbb{Z}$ . Then we can use the formula  $A^T J A = J$  to give

$$m'_{i,j} = \begin{pmatrix} a'_i & b'_i \end{pmatrix} J \begin{pmatrix} c'_j \\ d'_j \end{pmatrix} = \begin{pmatrix} a_i & b_i \end{pmatrix} A^T J A \begin{pmatrix} c_j \\ d_j \end{pmatrix} = m_{i,j},$$

as required.  $\square$

Lemma 6.4 shows that the map  $\tilde{\Phi}$  induces a function  $\mathrm{SL}_2(R) \backslash (\mathcal{P} \times \mathcal{P}) \rightarrow \mathbf{SL}_2$ . The next lemma demonstrates that this function is bijective.

**Lemma 6.5.** *The function  $\tilde{\Phi}: \mathcal{P} \times \mathcal{P} \rightarrow \mathbf{SL}_2$  is surjective. Furthermore, if  $\tilde{\Phi}(\gamma, \delta) = \tilde{\Phi}(\gamma', \delta')$ , then  $(\gamma', \delta') = (A\gamma, A\delta)$ , for some  $A \in \mathrm{SL}_2(R)$ .*

*Proof.* First we prove that  $\tilde{\Phi}$  is surjective. Let  $\mathbf{M}$  be a tame  $\mathrm{SL}_2$ -tiling over  $R$  and let  $(r_i)$  and  $(s_j)$  be the sequences in  $R$  associated to  $\mathbf{M}$  specified by Theorem 6.1. We define  $\gamma$  to be the path in  $\mathcal{P}$  with itinerary  $(r_i)$  and vertices  $(a_i, b_i)$  that satisfy  $(a_0, b_0) = (m_{0,0}, m_{0,1})$  and  $(a_1, b_1) = (m_{1,0}, m_{1,1})$ , and we define  $\delta$  to be the path with itinerary  $(s_j)$  and vertices  $(c_j, d_j)$  that satisfy  $(c_0, d_0) = (0, 1)$  and  $(c_1, d_1) = (-1, 0)$ . Now let  $\mathbf{M}'$  be the tame  $\mathrm{SL}_2$ -tiling  $\tilde{\Phi}(\gamma, \delta)$ , with entries  $m'_{i,j} = a_i d_j - b_i c_j$ . Then  $m'_{i,j} = m_{i,j}$ , for  $i, j = 0, 1$ . What is more, applying Lemma 3.2, we have

$$m'_{i-1,j} + m'_{i+1,j} = (a_{i-1} + a_{i+1})d_j - (b_{i-1} + b_{i+1})c_j = r_i a_i d_j - r_i b_i c_j = r_i m'_{i,j},$$

and similarly  $m'_{i,j-1} + m'_{i,j+1} = s_j m'_{i,j}$ . These recurrence relations specify  $\mathbf{M}'$  uniquely from  $m_{0,0}, m_{1,0}, m_{0,1}, m_{1,1}$ , so  $\mathbf{M}' = \mathbf{M}$ .

For the second part, suppose that  $\tilde{\Phi}(\gamma, \delta) = \tilde{\Phi}(\gamma', \delta')$ , and let  $\gamma, \delta, \gamma', \delta' \in \mathcal{P}$  have vertices  $(a_i, b_i), (c_j, d_j), (a'_i, b'_i)$ , and  $(c'_j, d'_j)$ . By replacing  $(\gamma, \delta)$  with  $(B\gamma, B\delta)$  and replacing  $(\gamma', \delta')$  with  $(B'\gamma', B'\delta')$ , for suitable matrices  $B, B' \in \mathrm{SL}_2(R)$ , and using Lemma 6.4, we can assume that  $(c_0, d_0) = (c'_0, d'_0) = (0, 1)$  and  $(c_1, d_1) = (c'_1, d'_1) = (-1, 0)$ . Let  $\mathbf{M} = \tilde{\Phi}(\gamma, \delta)$ . Then, as before, we can see that the itinerary  $(e_i)$  of  $\gamma$  satisfies  $m_{i-1,j} + m_{i+1,j} = e_i m_{i,j}$ , for  $i \in \mathbb{Z}$ , and similarly for the itinerary of  $\gamma'$ . From the uniqueness part of Theorem 6.1 it follows that the itineraries of  $\gamma$  and  $\gamma'$  are equal, and likewise so are those of  $\delta$  and  $\delta'$ . Hence  $\delta$  and  $\delta'$  are equal, by Lemma 3.3. Also, because  $(c_0, d_0) = (c'_0, d'_0) = (0, 1)$ , we have

$$a_0 = a_0 d_0 - b_0 c_0 = a'_0 d'_0 - b'_0 c'_0 = a'_0,$$

and similarly  $b_0 = b'_0$ ,  $a_1 = a'_1$ , and  $b_1 = b'_1$ . Therefore  $\gamma$  and  $\gamma'$  are equal also, as required.  $\square$

To facilitate the formal definition of the map  $\Phi_U$  from Theorem 1.3, we introduce an intermediate function  $\tilde{\Phi}_U: \mathcal{P}_U \times \mathcal{P}_U \rightarrow (U \times U) \backslash \mathbf{SL}_2$ . For this we define  $\tau_U: \mathbf{SL}_2 \rightarrow (U \times U) \backslash \mathbf{SL}_2$  to be the map that takes a tame  $\mathrm{SL}_2$ -tiling  $\mathbf{M}$  to its orbit under the action of  $U \times U$  on  $\mathbf{SL}_2$  (given by  $m_{i,j} \mapsto \lambda^{(-1)^i} \mu^{(-1)^j} m_{i,j}$ , for  $(\lambda, \mu) \in U \times U$ ). We recall that  $\pi_U: \mathcal{E}_R \rightarrow \mathcal{F}_{R,U}$  is the usual covering map. Then we define

$$\tilde{\Phi}_U(\pi_U(\gamma), \pi_U(\delta)) = \tau_U \tilde{\Phi}(\gamma, \delta), \quad \text{for } (\gamma, \delta) \in \mathcal{P} \times \mathcal{P}.$$

To see that  $\tilde{\Phi}_U$  is well-defined, first observe that, because  $\pi_U$  is a covering map, any pair in  $\mathcal{P}_U \times \mathcal{P}_U$  lifts to a pair in  $\mathcal{P} \times \mathcal{P}$ . Next, suppose that  $(\pi_U(\gamma), \pi_U(\delta)) = (\pi_U(\gamma'), \pi_U(\delta'))$ . Let  $\gamma$  have vertices  $(a_i, b_i)$  and  $\delta$  have vertices  $(c_j, d_j)$ ; then  $\tilde{\Phi}(\gamma, \delta)$  has entries  $m_{i,j} = a_i d_j - b_i c_j$ . By Lemma 3.1,  $\gamma'$  and  $\delta'$  have vertices  $\lambda^{(-1)^i} (a_i, b_i)$  and  $\mu^{(-1)^j} (c_j, d_j)$ , respectively, for some  $\lambda, \mu \in U$ , in which case  $\tilde{\Phi}(\gamma', \delta')$  has entries

$$m'_{i,j} = \lambda^{(-1)^i} \mu^{(-1)^j} (a_i d_j - b_i c_j) = \lambda^{(-1)^i} \mu^{(-1)^j} m_{i,j}.$$

Hence  $\tau_U \tilde{\Phi}(\gamma, \delta) = \tau_U \tilde{\Phi}(\gamma', \delta')$ , as required.

The next lemma proves an  $\mathrm{SL}_2(R)$ -invariance property of  $\tilde{\Phi}_U$  that has already been established for  $\tilde{\Phi}$  in Lemma 6.4.

**Lemma 6.6.** *The map  $\tilde{\Phi}_U$  satisfies  $\tilde{\Phi}_U(A\gamma, A\delta) = \tilde{\Phi}_U(\gamma, \delta)$ , for  $A \in \mathrm{SL}_2(R)$ .*

*Proof.* We choose  $(\tilde{\gamma}, \tilde{\delta}) \in \mathcal{P} \times \mathcal{P}$  with  $\pi_U(\tilde{\gamma}) = \gamma$  and  $\pi_U(\tilde{\delta}) = \delta$ . Recall from Proposition 2.4 that  $\pi_U A = A\pi_U$  (using the current terminology). Then  $\tilde{\Phi}_U(A\pi_U(\tilde{\gamma}), A\pi_U(\tilde{\delta})) = \tilde{\Phi}_U(\pi_U(A\tilde{\gamma}), \pi_U(A\tilde{\delta}))$ , so

$$\tilde{\Phi}_U(A\pi_U(\tilde{\gamma}), A\pi_U(\tilde{\delta})) = \tau_U \tilde{\Phi}(A\tilde{\gamma}, A\tilde{\delta}) = \tau_U \tilde{\Phi}(\tilde{\gamma}, \tilde{\delta}) = \tilde{\Phi}_U(\pi_U(\tilde{\gamma}), \pi_U(\tilde{\delta})),$$

where we have applied Lemma 6.4. □

From Lemma 6.6 we see that  $\tilde{\Phi}_U$  induces a function

$$\Phi_U: \mathrm{SL}_2(R) \backslash (\mathcal{P}_U \times \mathcal{P}_U) \longrightarrow (U \times U) \backslash \mathbf{SL}_2,$$

which maps the  $\mathrm{SL}_2(R)$ -orbit of a pair  $(\gamma, \delta)$  in  $\mathcal{P}_U \times \mathcal{P}_U$  to  $\tilde{\Phi}_U(\gamma, \delta)$ . This map is surjective, since  $\tilde{\Phi}$  is surjective. To complete the proof of Theorem 1.3 we have only to show that  $\Phi_U$  is injective.

Suppose then that  $\tilde{\Phi}_U(\pi_U(\gamma), \pi_U(\delta)) = \tilde{\Phi}_U(\pi_U(\gamma'), \pi_U(\delta'))$ , for paths  $\gamma, \delta, \gamma', \delta' \in \mathcal{P}$  with vertices  $(a_i, b_i)$ ,  $(c_j, d_j)$ ,  $(a'_i, b'_i)$ , and  $(c'_j, d'_j)$ . The  $\mathrm{SL}_2$ -tilings  $\mathbf{M}$  and  $\mathbf{M}'$  with entries  $m_{i,j} = a_i d_j - b_i c_j$  and  $m'_{i,j} = a'_i d'_j - b'_i c'_j$  lie in the same orbit of  $U \times U$ , so there exist  $\lambda, \mu \in U$  with  $m_{i,j} = \lambda^{(-1)^i} \mu^{(-1)^j} m'_{i,j}$ , for  $i, j \in \mathbb{Z}$ . Let  $\gamma'', \delta'' \in \mathcal{P}$  have vertices  $\lambda^{(-1)^i} (a'_i, b'_i)$  and  $\mu^{(-1)^j} (c'_j, d'_j)$ , respectively. Then  $(\pi_U(\gamma''), \pi_U(\delta'')) = (\pi_U(\gamma'), \pi_U(\delta'))$ , by Lemma 3.1, and the  $\mathrm{SL}_2$ -tiling  $\mathbf{M}''$  with entries  $m''_{i,j} = \lambda^{(-1)^i} a'_i \mu^{(-1)^j} d'_j - \lambda^{(-1)^i} b'_i \mu^{(-1)^j} c'_j = m_{i,j}$  coincides with  $\mathbf{M}$ . We can then apply Lemma 6.5 to see that there exists  $A \in \mathrm{SL}_2(R)$  with  $(\gamma'', \delta'') = (A\gamma, A\delta)$ . Consequently,  $(\pi_U(\gamma''), \pi_U(\delta'')) = (A\pi_U(\gamma), A\pi_U(\delta))$ , so  $\Phi_U$  is indeed injective. This completes the proof of Theorem 1.3.

## 7 Tame infinite friezes

As a stepping stone from  $\mathrm{SL}_2$ -tilings to friezes we look at infinite friezes, which have been considered by Baur, Parsons, and Tschabold [3] among others.

**Definition.** An infinite frieze over a ring  $R$  is a function  $\mathbf{F}: \{(i, j) \in \mathbb{Z}^2 : i \geq j\} \rightarrow R$  with entries  $m_{i,j} = \mathbf{F}(i, j)$  such that

- $m_{i,i} = 0$ , for  $i \in \mathbb{Z}$  (top row of zeros),
- $m_{i,j} m_{i+1,j+1} - m_{i,j+1} m_{i+1,j} = 1$ , for  $i > j$  (diamond rule).

The infinite frieze  $\mathbf{F}$  is *semiregular* if  $m_{i+1,i} = 1$ , for  $i \in \mathbb{Z}$  (second row of 1's), and *tame* if the usual 3-by-3 determinant 0 condition holds for  $\mathbf{F}$ , wherever it makes sense.

The second row of an infinite frieze can be calculated from any single entry using the diamond rule. This observation is encapsulated in the next lemma.

**Lemma 7.1.** *Let  $\mathbf{F}$  be a frieze or infinite frieze. Then  $m_{i+1,i} = \alpha^{(-1)^i}$ , for  $i \in \mathbb{Z}$ , where  $\alpha = m_{1,0}$ .*

*Proof.* Since  $\mathbf{F}$  is a frieze or infinite frieze we have  $m_{i,i-1} m_{i+1,i} - m_{i,i} m_{i+1,i-1} = 1$ , for each  $i \in \mathbb{Z}$ . But  $m_{i,i} = 0$ , so  $m_{i,i-1} m_{i+1,i} = 1$ . The equation  $m_{i+1,i} = \alpha^{(-1)^i}$  then follows by induction. □

Any tame infinite frieze can be extended to a tame  $\mathrm{SL}_2$ -tiling in the following manner.

**Lemma 7.2.** *Let  $\mathbf{F}$  be a tame infinite frieze and let  $\mathbf{M}: \mathbb{Z} \times \mathbb{Z} \rightarrow R$  (with entries  $m_{i,j}$ ) satisfy  $m_{i,j} = \mathbf{F}(i,j)$ , for  $i \geq j$ , and  $m_{j,i} = -\alpha^{(-1)^i - (-1)^j} m_{i,j}$ , for  $i, j \in \mathbb{Z}$ , where  $\alpha = m_{1,0}$ . Then  $\mathbf{M}$  is a tame  $\mathrm{SL}_2$ -tiling.*

*Proof.* Let us first check that  $\mathbf{M}$  is a function. Using the formula  $m_{j,i} = -\alpha^{(-1)^i - (-1)^j} m_{i,j}$  for  $i \geq j$  we can define  $\mathbf{M}(i,j)$  for  $i \leq j$ . It is straightforward to check that, with this definition, the formula  $m_{j,i} = -\alpha^{(-1)^i - (-1)^j} m_{i,j}$  continues to hold for  $i \leq j$  also. Hence  $\mathbf{M}$  is indeed a well-defined function.

Next we check that the diamond rule  $m_{i,j}m_{i+1,j+1} - m_{i,j+1}m_{i+1,j} = 1$  is satisfied for all  $i, j \in \mathbb{Z}$ . Certainly it is satisfied when  $i > j$ , by definition of a tame infinite frieze. If  $i < j$ , then one can check that the rule is satisfied by applying  $m_{j,i} = -\alpha^{(-1)^i - (-1)^j} m_{i,j}$ . Last, for  $i = j$ , we observe that  $m_{i,i+1} = -\alpha^{(-1)^{i+1} - (-1)^i} m_{i+1,i} = -\alpha^{-2(-1)^i} m_{i+1,i}$  and  $m_{i+1,i} = \alpha^{(-1)^i}$ , so

$$m_{i,i}m_{i+1,i+1} - m_{i,i+1}m_{i+1,i} = \alpha^{-2(-1)^i} m_{i+1,i}^2 = 1.$$

It remains to check that  $\mathbf{M}$  is tame; that is, we must check that  $\det A = 0$ , where

$$A = \begin{pmatrix} m_{i-1,j-1} & m_{i-1,j} & m_{i-1,j+1} \\ m_{i,j-1} & m_{i,j} & m_{i,j+1} \\ m_{i+1,j-1} & m_{i+1,j} & m_{i+1,j+1} \end{pmatrix},$$

for each  $i, j \in \mathbb{Z}$ . One can check that this is true if  $|i - j| > 1$  by using the tame property of  $\mathbf{F}$  and the relation  $m_{j,i} = -\alpha^{(-1)^i - (-1)^j} m_{i,j}$ . Suppose that  $i = j$ . Then each entry on the leading diagonal is 0, and a short calculation shows that  $\det A = m_{j+1,j-1} + m_{j-1,j+1} = 0$ . Suppose now that  $i = j + 1$ . Then  $m_{i,j}$  is a unit, by Lemma 7.1, so  $\det A = 0$ , by Lemma 6.2, using the diamond rule. The case  $i = j - 1$  can be handled similarly. Hence  $\mathbf{M}$  is a tame  $\mathrm{SL}_2$ -tiling, as required.  $\square$

We refer to the  $\mathrm{SL}_2$ -tiling  $\mathbf{M}$  obtained from the tame infinite frieze  $\mathbf{F}$  as the *extension* of  $\mathbf{F}$  to a tame  $\mathrm{SL}_2$ -tiling.

Theorem 1.3 taught us that pairs of bi-infinite paths can be used to classify tame  $\mathrm{SL}_2$ -tilings; here we will see that single bi-infinite paths can be used to classify infinite friezes. This observation was made by the first author for the usual Farey complex over the integers in [21]. We extend this to all rings and groups of units. Let us start with an elementary lemma.

**Lemma 7.3.** *Let  $(a, b)$  and  $(c, d)$  be vertices of  $\mathcal{E}_R$  with  $ad - bc = 0$ . Then  $(c, d) = \lambda(a, b)$ , for some  $\lambda \in R^\times$ .*

*Proof.* Since  $(a, b) \in \mathcal{E}_R$ , we have  $pa + qb = 1$ , for some  $p, q \in R$ . Let  $\lambda = pc + qd$ . Then

$$\lambda a = (pc + qd)a = pca + qbc = (pa + qb)c = c,$$

and similarly  $\lambda b = d$ , as required.  $\square$

The next lemma is central to our classification of tame infinite friezes using bi-infinite paths.

**Lemma 7.4.** *Let  $\mathbf{F}$  be a tame infinite frieze, and let  $\mathbf{M}$  be the extension of  $\mathbf{F}$  to a tame  $\mathrm{SL}_2$ -tiling. Then there exists a bi-infinite path  $\gamma$  in  $\mathcal{E}_R$  with vertices  $(a_i, b_i)$  such that*

$$m_{i,j} = \alpha^{(-1)^j} (a_j b_i - b_j a_i), \quad \text{for } i, j \in \mathbb{Z},$$

where  $\alpha = m_{1,0}$ .

*Proof.* By Lemma 6.5, there are paths  $\gamma, \delta \in \mathcal{P}$  with vertices  $(a_i, b_i)$  and  $(c_j, d_j)$  such that  $m_{i,j} = a_i d_j - b_i c_j$ . Since  $m_{i,i} = 0$ , we have that  $a_i d_i - b_i c_i = 0$ , for each  $i \in \mathbb{Z}$ . Lemma 7.3 tells us that there is  $\lambda_j \in R$  with  $(c_j, d_j) = \lambda_j (a_j, b_j)$ . From Lemma 7.1 we have  $m_{j+1,j} = \alpha^{(-1)^j}$ , so

$$\lambda_j = -\lambda_j (a_{j+1} b_j - b_{j+1} a_j) = -(a_{j+1} d_j - b_{j+1} c_j) = -m_{j+1,j} = -\alpha^{(-1)^j}.$$

Hence  $m_{i,j} = \alpha^{(-1)^j} (a_j b_i - b_j a_i)$ , as required.  $\square$

The next theorem shows that we can construct a tame infinite frieze uniquely from any suitable second row and any choice of third row whatsoever.

**Theorem 7.5.** *Given any bi-infinite sequence  $(r_i)$  in  $R$  and  $\alpha \in R^\times$  there is a unique tame infinite frieze with  $m_{i+1,i} = \alpha^{(-1)^i}$  and  $m_{i+1,i-1} = r_i$ , for  $i \in \mathbb{Z}$ .*

*Proof.* First we prove the existence of such an infinite frieze. By Lemma 3.3, there is a bi-infinite path  $\gamma$  in  $\mathcal{E}_R$  with itinerary  $(\alpha^{(-1)^i} r_i)$ ; let  $(a_i, b_i)$  be the vertices of this path and let  $\delta$  be the path with vertices  $-\alpha^{(-1)^j} (a_j, b_j)$ . Then  $\mathbf{M} = \tilde{\Phi}(\gamma, \delta)$  is a tame  $\mathrm{SL}_2$ -tiling with entries  $m_{i,j} = \alpha^{(-1)^j} (a_j b_i - b_j a_i)$ . Hence  $m_{i,i} = 0$ ,  $m_{i+1,i} = \alpha^{(-1)^i}$ , and

$$m_{i+1,i-1} = \alpha^{(-1)^i} (a_{i-1} b_{i+1} - b_{i-1} a_{i+1}) = r_i,$$

by definition of the itinerary of  $\gamma$ . Thus the restriction of  $\mathbf{M}$  to  $\{(i, j) \in \mathbb{Z}^2 : i \geq j\}$  is a tame infinite frieze with the required properties.

It remains to prove the uniqueness part of the theorem. Suppose then that  $\mathbf{F}$  is a tame infinite frieze with  $m_{i+1,i} = \alpha^{(-1)^i}$  and  $m_{i+1,i-1} = r_i$ , for  $i \in \mathbb{Z}$ , and let  $\mathbf{M}$  be the extension of  $\mathbf{F}$  to a tame  $\mathrm{SL}_2$ -tiling. From Lemma 7.4 we can find a bi-infinite path  $\gamma$  in  $\mathcal{E}_R$  with vertices  $(a_i, b_i)$  such that  $m_{i,j} = \alpha^{(-1)^j} (a_j b_i - b_j a_i)$ . Let  $(e_i)$  be the itinerary of  $\gamma$ . Then

$$e_i = a_{i-1} b_{i+1} - b_{i-1} a_{i+1} = \alpha^{(-1)^i} m_{i+1,i-1} = \alpha^{(-1)^i} r_i.$$

We can now apply Lemma 3.3 to see that  $\gamma$  is uniquely specified by its itinerary (itself determined by  $(r_i)$  and  $\alpha$ ) and an initial directed edge. However, up to  $\mathrm{SL}_2(R)$  equivalence we can choose any edge in  $\mathcal{E}_R$  as the initial directed edge, and the formula  $m_{i,j} = \alpha^{(-1)^j} (a_j b_i - b_j a_i)$  is preserved under the action of  $\mathrm{SL}_2(R)$ . It follows that  $\mathbf{M}$  is uniquely specified by  $(r_i)$  and  $\alpha$ .  $\square$

It is worthwhile highlighting the special case of Theorem 7.5 for semiregular infinite friezes (the  $\alpha = 1$  case). This result is known already (see [18, Theorem 1.15]), and indeed one can easily obtain Theorem 7.5 from this special case.

**Corollary 7.6.** *Given any bi-infinite sequence  $s$  in  $R$  there is a unique tame semiregular infinite frieze with quiddity sequence  $s$ .*

In the introduction we defined the map  $\tilde{\Psi}: \mathcal{P} \rightarrow \mathbf{SL}_2$  by the rule  $\tilde{\Psi}(\gamma) = \tilde{\Phi}(\gamma, -\gamma)$ . This map sends the bi-infinite path  $\gamma$  in  $\mathcal{E}_R$  with vertices  $(a_i, b_i)$  to the tame  $\mathbf{SL}_2$ -tiling  $\mathbf{M}$  over  $R$  with entries  $m_{i,j} = a_j b_i - b_j a_i$ . Observe that  $m_{i,i} = 0$  and  $m_{i+1,i} = 1$ , for  $i \in \mathbb{Z}$ , so  $\mathbf{M}$  is in fact (the extension of) a tame semiregular infinite frieze. Since  $\tilde{\Phi}(A\gamma, -A\gamma) = \tilde{\Phi}(\gamma, -\gamma)$ , for  $A \in \mathbf{SL}_2(R)$ , it follows that  $\tilde{\Psi}(A\gamma) = \tilde{\Psi}(\gamma)$ , so we obtain an induced map  $\Psi$  from  $\mathbf{SL}_2(R) \backslash \mathcal{P}$  to the set of tame semiregular infinite friezes. Our last result of this section is that this map is bijective.

**Theorem 7.7.** *The map  $\Psi$  induces a one-to-one correspondence between*

$$\mathbf{SL}_2(R) \backslash \left\{ \begin{array}{l} \text{bi-infinite} \\ \text{paths in } \mathcal{E}_R \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{tame semiregular infinite} \\ \text{friezes over } R \end{array} \right\}.$$

*Proof.* We recall from Theorem 1.3 that the map  $\Phi: \mathbf{SL}_2(R) \backslash (\mathcal{P} \times \mathcal{P}) \rightarrow \mathbf{SL}_2$  (where  $\Phi = \Phi_{\{1\}}$ ) is a one-to-one correspondence. A straightforward consequence is that  $\Psi$  is injective. All that remains is to prove that  $\Psi$  is surjective; however, this has been done already, in Lemma 7.4 for the case  $\alpha = 1$ .  $\square$

A version of this result for integer friezes was proven by the first author in [21, Theorem 1.3].

## 8 Tame friezes

In this section we prove Theorem 1.4. First, though, we prove (as promised in the introduction) that a tame frieze  $\mathbf{F}$  of width  $n$  has a unique extension to a tame  $\mathbf{SL}_2$ -tiling. We require a preparatory lemma, which is similar to Lemma 7.1.

**Lemma 8.1.** *Let  $\mathbf{F}$  be a frieze of width  $n$ . Then  $m_{i+n-1,i} = \beta^{(-1)^i}$ , for  $i \in \mathbb{Z}$ , where  $\beta = m_{n-1,0}$ .*

*Proof.* Since  $\mathbf{F}$  is a frieze we have  $m_{i+n-1,i} m_{i+n,i+1} - m_{i+n-1,i+1} m_{i+n,i} = 1$ , for each  $i \in \mathbb{Z}$ . But  $m_{i+n,i} = 0$ , so  $m_{i+n-1,i} m_{i+n,i+1} = 1$ , and the result follows by induction.  $\square$

Next we can state the extension result.

**Proposition 8.2.** *Let  $\mathbf{F}$  be a tame frieze of width  $n$  (with entries  $m_{i,j}$ ) and let  $\mathbf{M}: \mathbb{Z} \times \mathbb{Z} \rightarrow R$  be the function determined by the recurrence relations  $m_{i+n,j} = -\lambda^{(-1)^i} m_{i,j}$ , for  $i, j \in \mathbb{Z}$ , where  $\lambda = m_{1,0}/m_{n-1,0}$ . Then  $\mathbf{M}$  is a tame  $\mathbf{SL}_2$ -tiling. Furthermore,  $\mathbf{M}$  is the unique tame  $\mathbf{SL}_2$ -tiling that coincides with  $\mathbf{F}$  on  $0 \leq i - j \leq n$ .*

*Proof.* We define  $\alpha = m_{1,0}$  and  $\beta = m_{n-1,0}$ , so  $\lambda = \alpha/\beta$ . The recurrence relations  $m_{i+n,j} = -\lambda^{(-1)^i} m_{i,j}$  specify a well-defined function  $\mathbf{M}$  because  $m_{i,i} = m_{i+n,i} = 0$ , for  $i \in \mathbb{Z}$ .

Let us establish the diamond rule  $m_{i,j} m_{i+1,j+1} - m_{i,j+1} m_{i+1,j} = 1$ , for  $i, j \in \mathbb{Z}$ . We know that this holds for  $0 < i - j < n$  because  $\mathbf{F}$  is a tame frieze of width  $n$ . For  $i = j$  we have  $m_{i,i} = 0$ ,  $m_{i+1,i} = \alpha^{(-1)^i}$  by Lemma 7.1, and

$$m_{i,i+1} = -\lambda^{(-1)^i} m_{i+n,i+1} = -\lambda^{(-1)^i} \beta^{(-1)^{i+1}} = -\alpha^{(-1)^i},$$

where we have applied Lemma 8.1 to obtain  $m_{i+n,i+1} = \beta^{(-1)^{i+1}}$ . With these values we see that the diamond rule is indeed satisfied with  $i = j$ . We have now established the diamond rule for

$0 \leq i - j < n$ , and the recurrence relations  $m_{i+n,j} = -\lambda^{(-1)^i} m_{i,j}$  can then be applied to show that the rule is satisfied for all  $i, j \in \mathbb{Z}$ ; we omit the details.

We must check that  $\mathbf{M}$  is tame; that is, we must check that  $\det A = 0$ , where

$$A = \begin{pmatrix} m_{i-1,j-1} & m_{i-1,j} & m_{i-1,j+1} \\ m_{i,j-1} & m_{i,j} & m_{i,j+1} \\ m_{i+1,j-1} & m_{i+1,j} & m_{i+1,j+1} \end{pmatrix},$$

for each  $i, j \in \mathbb{Z}$ . This holds when  $1 < i - j < n - 1$  because  $\mathbf{F}$  is tame. It also holds whenever  $m_{i,j}$  is a unit, by Lemma 6.2 (and the diamond rule), so in particular it holds for  $i = j + 1$  and  $i = j + n - 1$ . When  $i = j$  each entry on the leading diagonal is 0, and a short calculation shows that  $\det A = m_{j+1,j-1} + m_{j-1,j+1} = 0$ . Therefore  $\det A = 0$  for  $0 \leq i - j < n$ , and the recurrence relations  $m_{i+n,j} = -\lambda^{(-1)^i} m_{i,j}$  can then be applied to show that  $\det A = 0$  for all  $i, j \in \mathbb{Z}$ ; again we omit the details.

It remains to prove the uniqueness part of the proposition. Suppose then that  $\mathbf{M}$  is any tame  $\mathbf{SL}_2$ -tiling that coincides with  $\mathbf{F}$  on  $0 \leq i - j \leq n$ . Then  $\mathbf{M}$  is a tame infinite frieze (since  $m_{i,i} = 0$ , for  $i \in \mathbb{Z}$ ). We can then invoke Theorem 7.5 to see that  $\mathbf{M}$  is uniquely specified by the second and third rows of  $\mathbf{F}$ , as required.  $\square$

There is an embedding of the collection  $\mathbf{FR}_n$  of tame friezes of width  $n$  into  $\mathbf{SL}_2$  that identifies a tame frieze  $\mathbf{F}$  with the unique tame  $\mathbf{SL}_2$ -tiling  $\mathbf{M}$  determined by Proposition 8.2. For convenience, we also write  $\mathbf{FR}_n$  for the image under this embedding; it comprises those tame  $\mathbf{SL}_2$ -tilings  $\mathbf{M}$  that satisfy  $m_{i,i} = 0$  and

$$m_{i+n,j} = -\lambda^{(-1)^i} m_{i,j}, \quad \text{for } i, j \in \mathbb{Z},$$

where  $\lambda \in R^\times$  (and in fact  $\lambda = m_{1,0}/m_{n-1,0}$ , by the uniqueness part of Proposition 8.2).

Let us now set about proving Theorem 1.4. We recall the map  $\tilde{\Psi}: \mathcal{P} \rightarrow \mathbf{SL}_2$  given by the rule  $\tilde{\Psi}(\gamma) = \tilde{\Phi}(\gamma, -\gamma)$ , which sends the bi-infinite path  $\gamma$  in  $\mathcal{E}_R$  with vertices  $v_i = (a_i, b_i)$  to the tame semiregular infinite frieze  $\mathbf{M}$  with entries  $m_{i,j} = a_j b_i - b_j a_i$ . Consider the restriction of  $\tilde{\Psi}$  to  $\mathcal{C}_n$ ; this set comprises bi-infinite paths  $\gamma$  with  $v_{i+n} = \lambda^{(-1)^i} v_i$ , for  $i \in \mathbb{Z}$ , where  $\lambda \in R^\times$ . Observe that

$$m_{i+n,j} = a_j b_{i+n} - b_j a_{i+n} = \lambda^{(-1)^i} (a_j b_i - b_j a_i) = \lambda^{(-1)^i} m_{i,j}.$$

Therefore  $\tilde{\Psi}(\gamma) \in \mathbf{FR}_n^*$  (the collection of tame semiregular friezes of width  $n$ ). Consequently, we have a map  $\tilde{\Psi}: \mathcal{C}_n \rightarrow \mathbf{FR}_n^*$ .

**Lemma 8.3.** *The map  $\tilde{\Psi}: \mathcal{C}_n \rightarrow \mathbf{FR}_n^*$  is surjective and  $\tilde{\Psi}(A\gamma) = \tilde{\Psi}(\gamma)$ , for  $A \in \mathbf{SL}_2(R)$ .*

*Proof.* First we prove that  $\tilde{\Psi}$  is surjective. Let  $\mathbf{M} \in \mathbf{FR}_n^*$ . By Lemma 7.4 we can find a bi-infinite path  $\gamma$  in  $\mathcal{E}_R$  with vertices  $v_i = (a_i, b_i)$  such that  $m_{i,j} = a_j b_i - b_j a_i$  (here  $\alpha = m_{1,0} = 1$  because  $\mathbf{M}$  is a semiregular frieze). Since  $m_{i+n,i} = 0$  we have  $v_{i+n} = \lambda_i v_i$ , for some  $\lambda_i \in R^\times$ , by Lemma 7.3. Observe that

$$\lambda_i \lambda_{i+1} = \lambda_i \lambda_{i+1} (a_i b_{i+1} - b_i a_{i+1}) = a_{i+n} b_{i+n+1} - b_{i+n} a_{i+n+1} = 1,$$

from which it follows that  $\lambda_i = \lambda^{(-1)^i}$ , where  $\lambda = \lambda_0$ . Therefore  $\gamma \in \mathcal{C}_n$ , so  $\tilde{\Psi}$  is indeed surjective.

The  $\mathbf{SL}_2(R)$ -invariance property follows immediately from the  $\mathbf{SL}_2(R)$ -invariance property of  $\tilde{\Phi}$  given in Lemma 6.4.  $\square$

Next, mirroring the procedure from Section 6, we define  $\tau_U: \mathbf{FR}_n^* \rightarrow U \backslash \mathbf{FR}_n^*$  to be the map that takes a tame semiregular frieze  $\mathbf{M}$  to its orbit under the action of  $U$  on  $\mathbf{FR}_n^*$  (given by  $m_{i,j} \mapsto \lambda^{(-1)^i + (-1)^j} m_{i,j}$ , for  $\lambda \in U$ ). Then we define  $\tilde{\Psi}_U: \mathcal{C}_{n,U} \rightarrow U \backslash \mathbf{FR}_n^*$  by

$$\tilde{\Psi}_U(\pi_U(\gamma)) = \tau_U \tilde{\Psi}(\gamma),$$

where  $\mathcal{C}_{n,U} = \pi_U(\mathcal{C}_n)$ . This is a well-defined function because if  $\pi_U(\gamma) = \pi_U(\gamma')$  – where  $\gamma$  and  $\gamma'$  have vertices  $(a_i, b_i)$  and  $\lambda^{(-1)^i} (a_i, b_i)$  for some  $\lambda \in U$  (see Lemma 3.1) – then  $\mathbf{M} = \tilde{\Psi}(\gamma)$  and  $\mathbf{M}' = \tilde{\Psi}(\gamma')$  satisfy

$$m'_{i,j} = a'_j b'_i - b'_j a'_i = \lambda^{(-1)^i + (-1)^j} m_{i,j}.$$

Thus  $\tau_U(\mathbf{M}') = \tau_U(\mathbf{M})$ .

Notice that  $\tilde{\Psi}_U(A\gamma) = \tilde{\Psi}_U(\gamma)$ , for  $A \in \mathrm{SL}_2(R)$ , by Lemma 8.3 and the equivariance of  $\pi_U$  under the action of  $\mathrm{SL}_2(R)$ . It follows that  $\tilde{\Psi}_U$  induces a map

$$\Psi_U: \mathrm{SL}_2(R) \backslash \mathcal{C}_{n,U} \rightarrow U \backslash \mathbf{FR}_n^*,$$

which is surjective since  $\tilde{\Psi}$  is surjective. To complete the proof of Theorem 1.5 we have only to show that  $\Psi_U$  is injective.

Suppose then that  $\tilde{\Psi}_U(\pi_U(\gamma)) = \tilde{\Psi}_U(\pi_U(\gamma'))$ , for  $\gamma, \gamma' \in \mathcal{C}_n$  with vertices  $(a_i, b_i)$  and  $(a'_i, b'_i)$ . Then  $\tau_U \tilde{\Psi}(\gamma) = \tau_U \tilde{\Psi}(\gamma')$ . By replacing  $(a'_i, b'_i)$  with  $\lambda^{(-1)^i} (a'_i, b'_i)$ , for some suitable unit  $\lambda \in U$  (which preserves  $\pi_U(\gamma')$ ), we can assume that in fact  $\tilde{\Psi}(\gamma) = \tilde{\Psi}(\gamma')$ . Then Lemma 6.5 tells us that  $\gamma' = A\gamma$ , for some  $A \in \mathrm{SL}_2(R)$ , so  $\pi_U(\gamma)$  and  $\pi_U(\gamma')$  lie in the same  $\mathrm{SL}_2(R)$ -orbit in  $\mathcal{C}_{n,U}$ . Therefore  $\Psi_U$  is indeed injective. This completes the proof of Theorem 1.4.

## 9 Tame regular friezes

In this section we prove Theorem 1.5, which says that the map  $\Psi$  induces a one-to-one correspondence between

$$\mathrm{SL}_2(R) \backslash \left\{ \begin{array}{l} \text{semiclosed paths of} \\ \text{length } n \text{ in } \mathcal{E}_R \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{tame regular friezes} \\ \text{over } R \text{ of width } n \end{array} \right\}.$$

Let us denote by  $\mathcal{S}_n$  the subcollection of  $\mathcal{C}_n$  of semiclosed paths of length  $n$ . A semiclosed path of length  $n$  considered as a bi-infinite path with vertices  $v_i$  satisfies  $v_{i+n} = -v_i$ , for  $i \in \mathbb{Z}$ . Let us denote by  $\mathbf{FR}_n^\dagger$  the subcollection of  $\mathbf{FR}_n^*$  of tame regular friezes of width  $n$ . By Lemma 8.3, the map  $\tilde{\Psi}: \mathcal{C}_n \rightarrow \mathbf{FR}_n^*$  is surjective. Suppose now that  $\gamma \in \mathcal{S}_n$  with vertices  $v_i = (a_i, b_i)$ . Then  $\mathbf{M} = \tilde{\Psi}(\gamma)$  is a tame semiregular frieze of width  $n$  and

$$m_{i+n-1,i} = a_i b_{i+n-1} - b_i a_{i+n-1} = -a_{i+n} b_{i+n-1} + b_{i+n} a_{i+n-1} = 1.$$

Hence  $\mathbf{M} \in \mathbf{FR}_n^\dagger$ .

Conversely, suppose that  $\mathbf{M} \in \mathbf{FR}_n^\dagger$  and  $\tilde{\Psi}(\gamma) = \mathbf{M}$ , where  $\gamma \in \mathcal{C}_n$ . Let  $\gamma$  have vertices  $v_i = (a_i, b_i)$ , where  $v_{i+n} = \lambda^{(-1)^i} v_i$ , for some  $\lambda \in R^\times$ . Then, because  $m_{n-1,0} = 1$ , we have

$$\lambda = \lambda m_{n-1,0} = \lambda(a_0 b_{n-1} - b_0 a_{n-1}) = a_n b_{n-1} - b_n a_{n-1} = -1.$$

Hence  $\gamma \in \mathcal{S}_n$ .

It follows that the restriction of  $\Psi$  to  $\mathrm{SL}_2(R) \backslash \mathcal{S}_n$  is a surjective map from  $\mathrm{SL}_2(R) \backslash \mathcal{S}_n$  onto  $\mathbf{FR}_n^\dagger$ . Now, we know from Theorem 1.4 that the map  $\Psi: \mathrm{SL}_2(R) \backslash \mathcal{E}_n \rightarrow \mathbf{FR}_n^*$  is a bijection, so it follows that the restriction map  $\Psi: \mathrm{SL}_2(R) \backslash \mathcal{S}_n \rightarrow \mathbf{FR}_n^\dagger$  is a bijection also. This completes the proof of Theorem 1.5.

## 10 Quiddity sequences of tame friezes

Here we prove Theorem 1.6, which says that any finite sequence in a finite ring  $R$  is the quiddity sequence for some tame semiregular frieze over  $R$ .

*Proof of Theorem 1.6.* Consider any finite sequence  $e_1, e_2, \dots, e_k$  in a finite ring  $R$ . We extend this to a periodic bi-infinite sequence by defining  $e_{k+i} = e_i$ , for  $i \in \mathbb{Z}$ . Let  $\gamma$  be any bi-infinite path in  $\mathcal{E}_R$  with itinerary  $(e_i)$ . As usual, we denote the vertices of  $\gamma$  by  $v_i = (a_i, b_i)$ . From Lemma 3.2 we have that

$$\begin{pmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{pmatrix} = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix} U_i, \quad \text{where } U_i = \begin{pmatrix} 0 & -1 \\ 1 & e_i \end{pmatrix},$$

for  $i \in \mathbb{Z}$ . Let  $g = U_1 U_2 \cdots U_k$  and let  $m$  be a positive integer such that  $g^m = I$ , the identity matrix in (the finite group)  $\mathrm{SL}_2(R)$ . We define  $n = mk$ ; then  $U_1 U_2 \cdots U_n = I$  and, more generally,  $U_i U_{i+1} \cdots U_{i+n-1} = I$ , for any  $i \in \mathbb{Z}$ . It follows that

$$\begin{pmatrix} a_{i+n-1} & a_{i+n} \\ b_{i+n-1} & b_{i+n} \end{pmatrix} = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix},$$

so  $v_{i+n} = v_i$ . As we saw in Section 8, the image of  $\gamma$  under  $\tilde{\Psi}$  is a tame semiregular frieze  $\mathbf{F}$  of width  $n$  with entries  $m_{i,j} = a_j b_i - b_j a_i$ . Observe that

$$m_{i+1, i-1} = a_{i-1} b_{i+1} - b_{i-1} a_{i+1} = e_i.$$

Thus  $e_1, e_2, \dots, e_k$  is a quiddity sequence for  $\mathbf{F}$ , as required.  $\square$

In contrast to Theorem 1.6, it is not true that any finite sequence  $e_1, e_2, \dots, e_k$  in a finite ring  $R$  is the quiddity sequence for some tame *regular* frieze over  $R$ . The simplest example to demonstrate this is the sequence with a single entry  $-1$  in the ring  $\mathbb{Z}/3\mathbb{Z}$ . We explore this no further here and instead refer the reader to [18, Theorem 1.15] for an alternative approach.

## 11 Enumerating friezes over finite fields

Here we prove Theorem 1.7, which says that the number of tame friezes of width  $n$  over a finite field  $R$  of size  $q$  is

$$\frac{(q-1)(q^{n-1} + (-1)^n)}{q+1}.$$

We use the Farey complex  $\mathcal{G}_R$ , which is the complete graph on  $q+1$  vertices. In this Farey complex any two equivalent vertices are in fact equal, so a path between equivalent vertices is a closed path.

*Proof of Theorem 1.7.* Let  $R$  denote the finite field of size  $q$ . The number of closed paths of length  $n$  in  $\mathcal{G}_R$  is  $q(q^{n-1} + (-1)^n)$ . This is a straightforward observation in graph theory, which can be proven by induction. Now, the group  $\mathrm{SL}_2(R)$  acts transitively on  $\mathcal{G}_R$ , and the kernel of this action is  $\{\pm I\}$ . It is a well-known and elementary observation that this group has size  $q(q^2 - 1)$ , so the quotient set

$$\mathrm{SL}_2(R) \setminus \left\{ \begin{array}{l} \text{closed paths of} \\ \text{length } n \text{ in } \mathcal{G}_R \end{array} \right\}$$

has size  $2(q^{n-1} + (-1)^n)/(q^2 - 1)$ . By Theorem 1.4, this is also the size of the set

$$R^\times \setminus \left\{ \begin{array}{l} \text{tame semiregular friezes} \\ \text{over } R \text{ of width } n \end{array} \right\}.$$

The group  $R^\times$  acts transitively on the collection of tame semiregular friezes of width  $n$  (by the rule  $m_{i,j} \mapsto \lambda^{(-1)^i + (-1)^j} m_{i,j}$ ) and the kernel of this action is  $\{\pm 1\}$ . It follows that there are  $(q^{n-1} + (-1)^n)/(q + 1)$  tame semiregular friezes over  $R$  of width  $n$ . Finally, as noted in the introduction, there is a  $(q - 1)$ -to-1 map from the full collection of tame friezes of width  $n$  to the subcollection of tame semiregular friezes given by  $m_{i,j} \mapsto \alpha^{(-1)^j} m_{i,j}$ , where  $\alpha = m_{1,0}$ . Consequently, there are  $(q - 1)(q^{n-1} + (-1)^n)/(q + 1)$  tame friezes over  $R$  of width  $n$ , as required.  $\square$

Next we recover a result of Morier-Genoud [19, Theorem 1] on the number of tame regular friezes of width  $n$ . Following Morier-Genoud, we frame the result using the notation of  $q$ -integers and  $q$ -binomial coefficients, with

$$[m]_{q^2} = \frac{q^{2m} - 1}{q^2 - 1} \quad \text{and} \quad \binom{m}{2}_q = \frac{(q^m - 1)(q^{m-1} - 1)}{(q - 1)(q^2 - 1)}.$$

**Theorem 11.1.** *The number  $\alpha_n$  of tame regular friezes of width  $n$  over the finite field  $R$  of size  $q$  is*

$$\alpha_n = \begin{cases} [m]_{q^2}, & \text{if } n = 2m + 1, \\ (q - 1) \binom{m}{2}_q & \text{if } n = 2m \text{ with } m \text{ even and } \mathrm{char} R \neq 2, \\ (q - 1) \binom{m}{2}_q + q^{m-1}, & \text{if } n = 2m \text{ with } m \text{ odd or } \mathrm{char} R = 2. \end{cases}$$

To prove Theorem 11.1, it is convenient to work with the Farey complex  $\mathcal{E}_R$ .

**Lemma 11.2.** *The number  $\mu(a, b)$  of paths of length 2 from  $(a, b)$  to  $(1, 0)$  in  $\mathcal{E}_R$ , where  $R$  is the finite field of size  $q$ , is given by*

$$\mu(a, b) = \begin{cases} 1, & \text{if } b \neq 0, \\ q, & \text{if } (a, b) = (-1, 0), \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Any path of length 2 from  $(a, b)$  to  $(1, 0)$  has the form

$$(a, b) \rightarrow (\lambda, -1) \rightarrow (1, 0),$$

where  $\lambda \in R$ . If  $b \neq 0$ , then there is only one such path, with  $\lambda = -(a + 1)b^{-1}$ . If  $(a, b) = (-1, 0)$ , then there are  $q$  such paths, since any value of  $\lambda \in R$  yields a path. Otherwise, if  $a \neq -1$  and  $b = 0$ , then no value of  $\lambda$  yields a path, so there are no paths of length 2 from  $(a, b)$  to  $(1, 0)$ .  $\square$

*Proof of Theorem 11.1.* For each  $n = 2, 3, \dots$  we define four classes of paths  $A_n$ ,  $B_n$ ,  $C_n$ , and  $D_n$  in  $\mathcal{E}_R$  as follows:

$D_n$  is the collection of all paths of length  $n$  with initial vertex  $(1, 0)$  and second vertex  $(0, 1)$ ,

$C_n$  is the subcollection of  $D_n$  of paths with final vertex  $(a, 0)$ , for some  $a \in R^\times$ ,

$B_n$  is the subcollection of  $C_n$  of paths with final vertex  $(1, 0)$ ,

$A_n$  is the subcollection of  $C_n$  of paths with final vertex  $(-1, 0)$ .

The collection  $D_n$  has size  $q^{n-1}$ . Let  $\lambda_n$  be the size of  $C_n$ . Under the covering map  $\pi_{R^\times} : \mathcal{E}_R \rightarrow \mathcal{G}_R$ , each element of  $C_n$  maps to a closed path in  $\mathcal{G}_R$ . Also, each closed path in  $\mathcal{G}_R$  has precisely  $q(q+1)$  preimages in  $C_n$  under  $\pi_{R^\times}$ . We noted earlier that there are  $q(q^{n-1} + (-1)^n)$  closed paths of length  $n$  in  $\mathcal{G}_R$ , so

$$\lambda_n = \frac{q^{n-1} + (-1)^n}{q+1}.$$

Next, recall that a semiclosed path in  $\mathcal{E}_R$  is a path with initial vertex  $v$  and final vertex  $-v$ , for some vertex  $v$  in  $\mathcal{E}_R$ . By Theorem 1.5 the number  $\alpha_n$  of tame regular friezes of width  $n$  over  $R$  is equal to the size of

$$\mathrm{SL}_2(R) \setminus \left\{ \begin{array}{l} \text{semiclosed paths of} \\ \text{length } n \text{ in } \mathcal{E}_R \end{array} \right\}.$$

Each semiclosed path of length  $n$  in  $\mathcal{E}_R$  is  $\mathrm{SL}_2(R)$ -equivalent to precisely one element of  $A_n$ ; hence  $\alpha_n$  is the size of  $A_n$ . Let  $\beta_n$  denote the size of  $B_n$ .

Consider the map  $A_n \rightarrow D_{n-2}$  that sends a path  $\langle v_0, v_1, \dots, v_n \rangle$  in  $A_n$  to  $\langle v_0, v_1, \dots, v_{n-2} \rangle$ . By Lemma 11.2, each element of  $D_{n-2} - C_{n-2}$  has exactly one preimage under this map, each element of  $B_{n-2}$  has exactly  $q$  preimages, and all other elements have no preimages. Consequently,

$$\alpha_n = (q^{n-3} - \lambda_{n-2}) + q\beta_{n-2},$$

and similarly one can see that

$$\beta_n = (q^{n-3} - \lambda_{n-2}) + q\alpha_{n-2}.$$

Suppose now that  $n$  is odd. In this case there is a bijection  $A_n \rightarrow B_n$  that sends

$$(1, 0) \rightarrow (0, 1) \rightarrow (a_2, b_2) \rightarrow (a_3, b_3) \rightarrow (a_4, b_4) \rightarrow (a_5, b_5) \rightarrow \dots \rightarrow (-1, 0)$$

to

$$(1, 0) \rightarrow (0, 1) \rightarrow (a_2, -b_2) \rightarrow (-a_3, b_3) \rightarrow (a_4, -b_4) \rightarrow (-a_5, b_5) \rightarrow \dots \rightarrow (1, 0).$$

Hence  $\alpha_n = \beta_n$ , so  $\alpha_n = (q^{n-3} - \lambda_{n-2}) + q\alpha_{n-2}$ . The formula  $\alpha_n = [m]_{q^2}$  can then be proven by induction.

Suppose instead that  $n$  is even. By subtracting the recurrence relations for  $\alpha_n$  and  $\beta_n$  we obtain  $\alpha_n - \beta_n = -q(\alpha_{n-2} - \beta_{n-2})$ . Since  $\alpha_2 = q$  and  $\beta_2 = 0$ , we see that  $\alpha_{2m} - \beta_{2m} = -(-q)^m$ . From this we obtain the revised recurrence relation

$$\alpha_{2m} = (q^{2m-3} - \lambda_{2m-2}) + q\alpha_{2m-2} - (-q)^{2m-1},$$

and the required formulas for  $\alpha_{2m}$  can then be established by induction.  $\square$

We note that the recent work [9] enumerates certain tame friezes over finite fields and  $\mathbb{Z}/N\mathbb{Z}$  for some values of  $N$ ; the friezes they consider are termed ‘quasiregular’ in Lemma 12.3, to follow.

## 12 Lifting $\mathrm{SL}_2$ -tilings

In this section we consider a third and final application of Theorems 1.3 to 1.5, on lifting  $\mathrm{SL}_2$ -tilings and friezes.

Let  $R$  be a ring and  $I$  an ideal in  $R$ ; then  $R/I$  is also a ring. The quotient map  $R \rightarrow R/I$  given by  $a \mapsto a + I$  induces a map  $\theta: \mathrm{SL}_2(R) \rightarrow \mathrm{SL}_2(R/I)$  by applying the quotient map to each entry, and it also induces a map  $\Theta$  from the collection of tame  $\mathrm{SL}_2$ -tilings over  $R$  to the collection of tame  $\mathrm{SL}_2$ -tilings over  $R/I$ . The first of three main results in this section follows.

**Theorem 12.1.** *For any ideal  $I$  in a ring  $R$ , the map  $\theta$  is surjective if and only if  $\Theta$  is surjective.*

The question of when  $\theta$  is surjective in general is complex. It is surjective if  $R$  is the ring of integers of a number field; on the other hand, there are examples such as [17, Example 13.5] for which the map is not surjective.

Let  $\rho: \mathcal{E}_R \rightarrow \mathcal{E}_{R/I}$  be the map with rule  $(a, b) \mapsto (a + I, b + I)$ . It is straightforward to prove that  $\rho$  is a graph homomorphism, in the sense that it maps vertices to vertices and preserves directed edges. It is not in general a covering map because it need not be locally bijective at vertices, and it may not be surjective. However, if  $\theta$  is surjective then  $\rho$  is surjective. This is straightforward to establish, for if  $u \rightarrow v$  is a directed edge in  $\mathcal{E}_{R/I}$ , then there is a matrix  $A \in \mathrm{SL}_2(R/I)$  with rows  $u$  and  $v$ . We can then find  $\tilde{A} \in \mathrm{SL}_2(R)$  with  $\theta(\tilde{A}) = A$ , in which case the rows  $\tilde{u}$  and  $\tilde{v}$  of  $\tilde{A}$  satisfy  $\rho(\tilde{u}) = u$  and  $\rho(\tilde{v}) = v$ .

In Section 5 we introduced principal congruence subgroups for  $\mathrm{SL}_2(\mathbb{Z})$ ; now we consider these groups for more general rings and ideals. The *principal congruence subgroup* of  $\mathrm{SL}_2(R)$  for the ideal  $I$  is the group

$$\Gamma_I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R) : a - 1, b, c, d - 1, \in I \right\}.$$

It is a normal subgroup of  $\mathrm{SL}_2(R)$ . It preserves the fibres of  $\rho$ , in the sense that the action of  $\Gamma_I$  on  $\mathcal{E}_R$  fixes each set  $\rho^{-1}(v)$ , for  $v \in \mathcal{E}_{R/I}$ . In fact,  $\Gamma_I$  acts transitively on each fibre.

**Lemma 12.2.** *The group  $\Gamma_I$  acts transitively on each fibre of  $\rho$ .*

*Proof.* Suppose that  $\rho(a, b) = \rho(c, d)$ ; we will prove that there exists an element of  $\Gamma_I$  that maps  $(a, b)$  to  $(c, d)$ .

Consider first the case  $(a, b) = (1, 0)$ . Let us choose  $x, y \in R$  such that  $cx - dy = 1$  and define

$$A = \begin{pmatrix} c & y(1 - c) \\ d & 1 + x(1 - c) \end{pmatrix}.$$

Then  $c - 1, y(1 - c), d, x(1 - c) \in I$  and  $\det A = c + (1 - c)(cx - dy) = 1$ , so  $A \in \Gamma_I$ . Also,  $A(1, 0) = (c, d)$ , as required.

For the general case, first choose  $B \in \mathrm{SL}_2(R)$  with  $B(a, b) = (1, 0)$ . Then we can find  $A \in \Gamma_I$  with  $AB(a, b) = B(c, d)$ , so  $B^{-1}AB(a, b) = (c, d)$ . Since  $\Gamma_I$  is a normal subgroup of  $\mathrm{SL}_2(R)$  we have found the required element of  $\Gamma_I$  that maps  $(a, b)$  to  $(c, d)$ .  $\square$

We will use Lemma 12.2 to prove that bi-infinite paths lift from  $\mathcal{E}_{R/I}$  to  $\mathcal{E}_R$  under  $\rho$ .



Evidently this does not lift to a tame  $\mathrm{SL}_2$ -tiling over  $R$  under  $\Theta$ .

Suppose now that  $\theta$  is surjective. Let  $\mathbf{M}$  be a tame  $\mathrm{SL}_2$ -tiling over  $R/I$ . Choose bi-infinite paths  $\gamma$  and  $\delta$  in  $\mathcal{E}_{R/I}$  with  $\tilde{\Phi}_{R/I}(\gamma, \delta) = \mathbf{M}$ . By Lemma 12.3 there are bi-infinite paths  $\tilde{\gamma}$  and  $\tilde{\delta}$  in  $\mathcal{E}_{R/I}$  with  $\rho(\tilde{\gamma}, \tilde{\delta}) = (\gamma, \delta)$ . Now let  $\tilde{\mathbf{M}} = \tilde{\Phi}_R(\tilde{\gamma}, \tilde{\delta})$ . Then

$$\Theta(\tilde{\mathbf{M}}) = \Theta(\tilde{\Phi}_R(\tilde{\gamma}, \tilde{\delta})) = \tilde{\Phi}_{R/I}(\rho(\tilde{\gamma}, \tilde{\delta})) = \mathbf{M}.$$

Hence  $\Theta$  is surjective, as required.  $\square$

For the remainder of the paper we restrict our attention to the case  $R = \mathbb{Z}$  and  $I = N\mathbb{Z}$ , where  $N > 1$ , which we considered already in Section 5. It is well known, and easy to establish, that the map  $\theta: \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$  is surjective, so Theorem 12.1 tells us that any tame  $\mathrm{SL}_2$ -tiling over  $\mathbb{Z}/N\mathbb{Z}$  lifts to a tame  $\mathrm{SL}_2$ -tiling over  $\mathbb{Z}$ . In fact, we will prove that any tame  $\mathrm{SL}_2$ -tiling over  $\mathbb{Z}/N\mathbb{Z}$  lifts to a *positive integer*  $\mathrm{SL}_2$ -tiling over  $\mathbb{Z}$ .

**Theorem 12.4.** *Given any tame  $\mathrm{SL}_2$ -tiling  $\mathbf{M}$  over  $\mathbb{Z}/N\mathbb{Z}$  there is a tame  $\mathrm{SL}_2$ -tiling  $\tilde{\mathbf{M}}$  with positive integer entries for which  $\Theta(\tilde{\mathbf{M}}) = \mathbf{M}$ .*

In proving this theorem we write  $\mathcal{E}_N$  in place of the more cumbersome  $\mathcal{E}_{\mathbb{Z}/N\mathbb{Z}}$ , just as we have been writing  $\mathcal{F}_N$  in place of  $\mathcal{F}_{\mathbb{Z}/N\mathbb{Z}}$ .

**Lemma 12.5.** *Let  $u \rightarrow v$  be a directed edge in  $\mathcal{E}_N$ , and let  $(a, b) \in \mathcal{E}_{\mathbb{Z}}$  satisfy  $\rho(a, b) = u$ ,  $b > 0$ , and  $-1 < a/b < 1$ . Then we can find a directed edge  $(a, b) \rightarrow (c, d)$  in  $\mathcal{E}_{\mathbb{Z}}$  with  $\rho(c, d) = v$ ,  $d > 0$ , and  $-1 < c/d < a/b$ .*

*Proof.* Since  $\theta: \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$  is surjective, we can certainly find a directed edge  $\tilde{u} \rightarrow \tilde{v}$  in  $\mathcal{E}_{\mathbb{Z}}$  that is mapped to  $u \rightarrow v$  by  $\rho$ . From Lemma 12.2, the principal congruence subgroup  $\Gamma_N$  acts transitively on  $\rho^{-1}(u)$ , so there exists  $A \in \Gamma_N$  with  $A\tilde{u} = (a, b)$ . Let  $(x, y) = A\tilde{v}$ . Then  $(a, b) \rightarrow (x, y)$  is a directed edge in  $\mathcal{E}_{\mathbb{Z}}$  and  $\rho(x, y) = v$ . Let us define  $(c, d) = (x + kNa, y + kNb)$ , where  $k$  is a positive integer chosen to be sufficiently large that  $d > 0$  and  $c/d > -1$ . Observe that  $(a, b) \rightarrow (c, d)$  is also a directed edge in  $\mathcal{E}_{\mathbb{Z}}$  and  $\rho(c, d) = v$ . Furthermore,

$$\frac{a}{b} - \frac{c}{d} = \frac{1}{bd}(ad - bc) = \frac{1}{bd}(ay - bx) = \frac{1}{bd} > 0.$$

Hence  $(c, d)$  has all the required properties.  $\square$

From this we deduce the following stronger version of Lemma 12.3.

**Lemma 12.6.** *For any bi-infinite path  $\gamma$  in  $\mathcal{E}_N$  there is a bi-infinite path  $\tilde{\gamma}$  in  $\mathcal{E}_{\mathbb{Z}}$  such that  $\rho(\tilde{\gamma}) = \gamma$  and such that the vertices  $(a_i, b_i)$  of  $\tilde{\gamma}$  satisfy  $b_i > 0$ , for  $i \in \mathbb{Z}$ , and*

$$-1 < \cdots < \frac{a_2}{b_2} < \frac{a_1}{b_1} < \frac{a_0}{b_0} < \frac{a_{-1}}{b_{-1}} < \frac{a_{-2}}{b_{-2}} < \cdots < 1.$$

*Proof.* It is straightforward to choose a preimage  $(a_0, b_0)$  of the zeroth vertex of  $\gamma$  such that  $b_0 > 0$  and  $-1 < a_0/b_0 < 1$ . We can then apply Lemma 12.5 and a recursive argument to construct suitable vertices  $(a_i, b_i)$ , for  $i \geq 0$ . Then, by changing the sign of  $a_0$  and applying a similar argument, we can extend the sequence to  $i \leq 0$ .  $\square$

We can now prove Theorem 12.4.

*Proof of Theorem 12.4.* We begin by choosing bi-infinite paths  $\gamma$  and  $\delta$  in  $\mathcal{E}_N$  with  $\tilde{\Phi}_{\mathbb{Z}/N\mathbb{Z}}(\gamma, \delta) = \mathbf{M}$ . By Lemma 12.6 there are paths  $\tilde{\gamma}$  and  $\tilde{\delta}$  in  $\mathcal{E}_{\mathbb{Z}}$  such that  $\rho(\tilde{\gamma}, \tilde{\delta}) = (\gamma, \delta)$  and such that the vertices  $(a_i, b_i)$  and  $(c_j, d_j)$  of  $\tilde{\gamma}$  and  $\tilde{\delta}$  satisfy  $b_i, d_j > 0$ , for  $i, j \in \mathbb{Z}$ , and

$$-1 < \dots < \frac{c_2}{d_2} < \frac{c_1}{d_1} < \frac{c_0}{d_0} < \frac{c_{-1}}{d_{-1}} < \frac{c_{-2}}{d_{-2}} < \dots < 1$$

and

$$N - 1 < \dots < \frac{a_2}{b_2} < \frac{a_1}{b_1} < \frac{a_0}{b_0} < \frac{a_{-1}}{b_{-1}} < \frac{a_{-2}}{b_{-2}} < \dots < N + 1.$$

(Replace  $(a_i, b_i)$  with  $(a_i + Nb_i, b_i)$  for the second set of inequalities.) Now let  $\tilde{\mathbf{M}} = \tilde{\Phi}_{\mathbb{Z}}(\tilde{\gamma}, \tilde{\delta})$ . Then, as before, we have  $\Theta(\tilde{\mathbf{M}}) = \mathbf{M}$ . What is more, the entries  $\tilde{m}_{i,j}$  of  $\tilde{\mathbf{M}}$  satisfy

$$\tilde{m}_{i,j} = a_i d_j - b_i c_j = b_i d_j \left( \frac{a_i}{b_i} - \frac{c_j}{d_j} \right) > 0,$$

as required.  $\square$

Theorems 12.1 and 12.4 tell us that every tame  $\mathrm{SL}_2$ -tiling over  $\mathbb{Z}/N\mathbb{Z}$  lifts to a tame  $\mathrm{SL}_2$ -tiling over  $\mathbb{Z}$  and in fact the lifted  $\mathrm{SL}_2$ -tiling can be chosen with positive entries. We learned in the introduction that a similar statement cannot be made for friezes. Here we will prove Theorem 1.8, which tells us exactly when a tame frieze over  $\mathbb{Z}/N\mathbb{Z}$  lifts to a tame frieze over  $\mathbb{Z}$ .

For this purpose it is more convenient to use the Farey complexes  $\mathcal{F}_N$  and  $\mathcal{F}_{\mathbb{Z}}$  rather than  $\mathcal{E}_N$  and  $\mathcal{E}_{\mathbb{Z}}$ , because  $\mathcal{F}_N$  is a surface complex, as we saw in Section 5, and  $\mathcal{E}_N$  is not. We recall the notation  $\bar{a}$  from Section 5 for the class of integers congruent to  $a$  modulo  $n$ . With this notation we have  $\rho(a, b) = (\bar{a}, \bar{b})$ . Notice that  $\rho$  maps  $\pm(a, b)$  to  $\pm(\bar{a}, \bar{b})$ , so we obtain a map from  $\mathcal{F}_{\mathbb{Z}}$  to  $\mathcal{F}_N$ , which we also denote by  $\rho$ . This preserves edges and triangles. Furthermore, if there is an edge between  $a/b$  and  $c/d$  in  $\mathcal{F}_{\mathbb{Z}}$  (where we have switched to fractional notation  $a/b$  in place of  $\pm(a, b)$ ), then the other vertices of the two triangles incident to this edge are  $(a+b)/(c+d)$  and  $(a-b)/(c-d)$ . Under  $\rho$  these are mapped to the vertices of the two triangles incident to the edge between  $\rho(a/b)$  and  $\rho(c/d)$ . (Unless  $n = 2$ , in which case they are mapped to the single common neighbour of  $\rho(a/b)$  and  $\rho(c/d)$ .)

We recall from the introduction that a (finite) closed path in a Farey complex  $\mathcal{F}_R$  is said to be strongly contractible if it can be transformed to a point by applying a finite number of the following two elementary homotopies.

(E1) Replace a subpath  $\langle v, u, v \rangle$  with the subpath  $\langle v \rangle$ .

(E2) Replace a subpath  $\langle u, v, w \rangle$  with  $\langle u, w \rangle$ , where  $u, v$ , and  $w$  are mutually adjacent.

The property of being strongly contractible is preserved under the action of  $\mathrm{SL}_2(R)$ , in the sense that if  $\gamma$  is strongly contractible then so is  $A\gamma$ , where  $A \in \mathrm{SL}_2(R)$ . Not all closed paths are strongly contractible, as Figure 1.10 demonstrates; however, they are in  $\mathcal{F}_{\mathbb{Z}}$ .

**Lemma 12.7.** *All closed paths in  $\mathcal{F}_{\mathbb{Z}}$  are strongly contractible.*

*Proof.* Suppose to the contrary that some closed paths in  $\mathcal{F}_{\mathbb{Z}}$  are not strongly contractible. Then there exists a closed path  $\gamma = \langle v_0, v_1, \dots, v_m \rangle$ , where  $m \geq 2$ , to which neither (E1) nor (E2) can be applied. We will demonstrate that this leads to a contradiction.

It is convenient to assume that  $v_0 = \infty$ , which can be achieved by applying an element of  $\mathrm{SL}_2(\mathbb{Z})$  to  $\gamma$ .

Let us choose coprime pairs of integers  $(a_i, b_i)$  with  $v_i = a_i/b_i$ , for  $i = 1, 2, \dots, m$  (and  $(a_0, b_0) = (1, 0)$ ). The neighbours in  $\mathcal{F}_{\mathbb{Z}}$  of  $a_i/b_i$  have the form  $(\lambda a_i - a_{i-1})/(\lambda b_i - b_{i-1})$ , for  $\lambda \in \mathbb{Z}$ . Hence  $a_{i+1} = \lambda_i a_i - a_{i-1}$  and  $b_{i+1} = \lambda_i b_i - b_{i-1}$ , for some  $\lambda_i \in \mathbb{Z}$ , where  $1 \leq i < m$ . It cannot be that  $\lambda_i = 0$ , for if that were so then  $v_{i+1}$  would equal  $v_{i-1}$  and we could apply (E1) to  $\gamma$ . Nor can we have  $\lambda_i = \pm 1$ , for in these cases  $v_{i-1}$ ,  $v_i$ , and  $v_{i+1}$  are mutually adjacent, so we could apply (E2) to  $\gamma$ . Thus  $|\lambda_i| \geq 2$ . As a consequence,

$$|b_{i+1}| = |\lambda_i b_i - b_{i-1}| \geq |b_i| + (|\lambda_i| - 1)|b_{i-1}|.$$

From this inequality it follows that the sequence  $|b_0|, |b_1|, \dots$  is increasing. Hence  $b_m \neq 0$ , so  $v_m \neq v_0$ , which is the desired contradiction. Therefore all closed paths in  $\mathcal{F}_{\mathbb{Z}}$  are strongly contractible after all.  $\square$

Suppose that  $\tilde{\gamma}_2$  is a closed path in  $\mathcal{F}_{\mathbb{Z}}$  obtained from another closed path  $\tilde{\gamma}_1$  by applying one of (E1) or (E2). Let  $\gamma_1 = \rho(\tilde{\gamma}_1)$  and  $\gamma_2 = \rho(\tilde{\gamma}_2)$ . Since  $\rho$  preserves edges and triangles, we can see that  $\gamma_2$  is obtained from  $\gamma_1$  by applying one of (E1) or (E2). From this observation we deduce the following lemma.

**Lemma 12.8.** *Let  $\tilde{\gamma}$  be a closed path in  $\mathcal{F}_{\mathbb{Z}}$  and let  $\gamma = \rho(\tilde{\gamma})$ . Then  $\gamma$  is strongly contractible in  $\mathcal{F}_N$ .*

*Proof.* By Lemma 12.7,  $\tilde{\gamma}$  is strongly contractible, so there is a finite sequence of elementary homotopies from  $\tilde{\gamma}$  to a single point. Since these elementary homotopies are preserved under  $\rho$  we see that there is a finite sequence of elementary homotopies from  $\gamma$  to a single point. Hence  $\gamma$  is strongly contractible in  $\mathcal{F}_N$ .  $\square$

In fact the converse to this lemma holds.

**Lemma 12.9.** *Suppose that  $\gamma$  is a strongly contractible closed path in  $\mathcal{F}_N$ . Then there exists a closed path  $\tilde{\gamma}$  in  $\mathcal{F}_{\mathbb{Z}}$  with  $\rho(\tilde{\gamma}) = \gamma$ .*

*Proof.* We prove this by induction on the length of  $\gamma$ . Observe first that any closed path of length 0 (a single point) is strongly contractible and lifts to a closed path of length 0 in  $\mathcal{F}_{\mathbb{Z}}$ .

Suppose now that each strongly contractible closed path of length  $k$  in  $\mathcal{F}_N$  lifts to a closed path of length  $k$  in  $\mathcal{F}_{\mathbb{Z}}$ , for  $k = 0, 1, \dots, m$ . Let  $\gamma$  be a strongly contractible closed path of length  $m + 1$  in  $\mathcal{F}_N$ . Since  $\gamma$  is strongly contractible, we can apply an elementary homotopy to  $\gamma$  to obtain another strongly contractible closed path  $\gamma'$ , which has length either  $m - 1$  or  $m$ , depending on whether (E1) or (E2) was applied. Let us suppose that (E2) was applied (the other case is handled similarly). Then we can write  $\gamma' = \langle v_0, v_1, \dots, v_m \rangle$ , where  $v_m = v_0$ . The path  $\gamma$  differs from  $\gamma'$  in that there is an additional vertex  $v$  inserted between  $v_{j-1}$  and  $v_j$ , where  $1 \leq j \leq m$ . By the induction hypothesis, there is a lift  $\tilde{\gamma}' = \langle \tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_m \rangle$  of  $\gamma'$  to  $\mathcal{F}_{\mathbb{Z}}$ , where  $\tilde{v}_m = \tilde{v}_0$ . Let  $\tilde{v}$  be the unique vertex in  $\mathcal{F}_{\mathbb{Z}}$  that is adjacent to both  $\tilde{v}_{j-1}$  and  $\tilde{v}_j$  and satisfies  $\rho(\tilde{v}) = v$  (in the exceptional case  $N = 2$  there are two choices for  $\tilde{v}$ ). We define  $\tilde{\gamma}$  to be the closed path obtained from  $\tilde{\gamma}'$  by inserting  $\tilde{v}$  between  $\tilde{v}_{j-1}$  and  $\tilde{v}_j$ . Then  $\rho(\tilde{\gamma}) = \gamma$ , and this completes the argument by induction.  $\square$

In Section 8 we introduced the surjective map  $\tilde{\Psi}_U$  from the set  $\mathcal{C}_{n,U}$  of paths of length  $n$  between equivalent vertices in  $\mathcal{F}_{R,U}$  to the set  $U \backslash \mathbf{FR}_n^*$ , where  $\mathbf{FR}_n^*$  is the collection of tame semiregular friezes over  $R$  of width  $n$ , and where  $U$  acts on  $\mathbf{FR}_n^*$  by the rule  $m_{i,j} \mapsto \lambda^{(-1)^i + (-1)^j} m_{i,j}$ , for  $\lambda \in U$ . Here we restrict our attention to  $U = \{\pm 1\}$ , in which case the action of  $U$  is trivial, so the image of  $\tilde{\Psi}_U$  is  $\mathbf{FR}_n^*$ . We will describe a semiregular frieze as *quasiregular* if its second-last row comprises all 1s or all  $-1$ s (that is,  $m_{i+n-1,i}$  is equal to (one of) 1 or  $-1$  for  $i \in \mathbb{Z}$ ).

**Lemma 12.10.** *Let  $\mathbf{F} = \tilde{\Psi}_U(\gamma)$ , for  $U = \{\pm 1\}$ , where  $\gamma \in \mathcal{C}_{n,U}$  and  $\mathbf{F} \in \mathbf{FR}_n^*$ . Then  $\gamma$  is a closed path if and only if  $\mathbf{F}$  is quasiregular.*

*Proof.* Let  $\tilde{\gamma}$  be a lift of  $\gamma$  to  $\mathcal{C}_n$ . Then  $\tilde{\gamma} = \langle v_0, v_1, \dots, v_n \rangle$ , where  $v_i = (a_i, b_i)$  and  $v_n = \lambda v_0$  for  $\lambda \in R^\times$ . Using the formula  $v_{i+n} = \lambda^{(-1)^i} v_i$  for the extension of  $\gamma$  to a bi-infinite path, we see that the second-last row of  $\mathbf{F}$  has entries

$$m_{i+n-1,i} = a_i b_{i+n-1} - b_i a_{i+n-1} = \lambda^{(-1)^{i-1}} (a_i b_{i-1} - b_i a_{i-1}) = -\lambda^{(-1)^{i-1}}.$$

Therefore  $\mathbf{F}$  is quasiregular if and only if  $\lambda \in \{\pm 1\}$ . The result follows, because  $\lambda \in \{\pm 1\}$  if and only if  $\gamma$  is a closed path in  $\mathcal{F}_R$ .  $\square$

We denote by  $\tilde{\Psi}_{\mathbb{Z}}$  and  $\tilde{\Psi}_N$  the two versions of the map  $\tilde{\Psi}_{\{\pm 1\}}$  for the rings  $R = \mathbb{Z}$  and  $R = \mathbb{Z}/N\mathbb{Z}$ , respectively. With this terminology, it is straightforward to check that the following diagram commutes.

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{closed paths of} \\ \text{length } n \text{ in } \mathcal{F}_{\mathbb{Z}} \end{array} \right\} & \xrightarrow{\tilde{\Psi}_{\mathbb{Z}}} & \left\{ \begin{array}{l} \text{tame quasiregular friezes} \\ \text{over } \mathbb{Z} \text{ of width } n \end{array} \right\} \\ \downarrow \rho & & \downarrow \Theta \\ \left\{ \begin{array}{l} \text{closed paths of} \\ \text{length } n \text{ in } \mathcal{F}_N \end{array} \right\} & \xrightarrow{\tilde{\Psi}_N} & \left\{ \begin{array}{l} \text{tame quasiregular friezes} \\ \text{over } \mathbb{Z}/N\mathbb{Z} \text{ of width } n \end{array} \right\} \end{array}$$

We are now ready to prove Theorem 1.8.

*Proof of Theorem 1.8.* Suppose first that  $\mathbf{F}$  is a tame semiregular frieze of width  $n$  over  $\mathbb{Z}/N\mathbb{Z}$  that lifts to a tame frieze  $\tilde{\mathbf{F}}$  of width  $n$  over  $\mathbb{Z}$ . Then  $\tilde{\mathbf{F}}$  is quasiregular, since it is defined over  $\mathbb{Z}$ . By Lemma 12.10, we can find a closed path  $\tilde{\gamma}$  of length  $n$  in  $\mathcal{F}_{\mathbb{Z}}$  with  $\tilde{\Psi}_{\mathbb{Z}}(\tilde{\gamma}) = \tilde{\mathbf{F}}$ . Let  $\gamma = \rho(\tilde{\gamma})$ . Then  $\gamma$  is strongly contractible, by Lemma 12.8. Also, we have

$$\tilde{\Psi}_N(\gamma) = \tilde{\Psi}_N(\rho(\tilde{\gamma})) = \Theta(\tilde{\Psi}_{\mathbb{Z}}(\tilde{\gamma})) = \mathbf{F}.$$

Thus  $\gamma$  is a strongly contractible closed path corresponding to  $\mathbf{F}$ . Furthermore, any other path corresponding to  $\mathbf{F}$  is also strongly contractible and closed because these properties are preserved under the action of  $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ .

For the converse, suppose that  $\gamma$  is a strongly contractible closed path corresponding to a tame semiregular frieze  $\mathbf{F}$ . Then  $\tilde{\Psi}_N(\gamma) = \mathbf{F}$ . By Lemma 12.9 there is closed path  $\tilde{\gamma}$  in  $\mathcal{F}_{\mathbb{Z}}$  with  $\rho(\tilde{\gamma}) = \gamma$ . Let  $\tilde{\mathbf{F}} = \tilde{\Psi}_{\mathbb{Z}}(\tilde{\gamma})$ . Then

$$\Theta(\tilde{\mathbf{F}}) = \Theta(\tilde{\Psi}_{\mathbb{Z}}(\tilde{\gamma})) = \tilde{\Psi}_N(\rho(\tilde{\gamma})) = \mathbf{F},$$

so  $\mathbf{F}$  does indeed lift to a tame frieze  $\tilde{\mathbf{F}}$  of width  $n$  over  $\mathbb{Z}$ , as required.  $\square$

## References

- [1] I. Assem, C. Reutenauer, and D. Smith, *Friezes*, Adv. Math. **225** (2010), no. 6, 3134–3165.
- [2] M. D. Baker and A. W. Reid, *Principal congruence link complements*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014), no. 5, 1063–1092.
- [3] K. Baur, M. J. Parsons, and M. Tschabold, *Infinite friezes*, European J. Combin. **54** (2016), 220–237.
- [4] A. F. Beardon, M. Hockman, and I. Short, *Geodesic continued fractions*, Michigan Math. J. **61** (2012), no. 1, 133–150.
- [5] G. Bini and F. Flamini, *Finite commutative rings and their applications*, The Kluwer International Series in Engineering and Computer Science, vol. 680, Kluwer Academic Publishers, Boston, MA, 2002.
- [6] D. J. Collins, R. I. Grigorchuk, P. F. Kurchanov, and H. Zieschang, *Combinatorial group theory and applications to geometry*, Springer-Verlag, Berlin, 1998.
- [7] J. H. Conway and H. S. M. Coxeter, *Triangulated polygons and frieze patterns*, Math. Gaz. **57** (1973), 87–94, 175–183.
- [8] H. S. M. Coxeter, *Frieze patterns*, Acta Arith. **18** (1971), 297–310.
- [9] M. Cuntz and F. Mabilat, *Comptage des quiddités sur les corps finis et sur quelques anneaux  $\mathbb{Z}/N\mathbb{Z}$*  (2023), available at <https://arxiv.org/abs/2304.03071>.
- [10] M. Cuntz, T. Holm, and C. Pagano, *Frieze patterns over algebraic numbers* (2023), available at <https://arxiv.org/abs/2306.12148>.
- [11] F. Diamond and J. Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005.
- [12] A. Felikson, O. Karpenkov, K. Serhiyenko, and P. Tumarkin, *3D Farey graph, lambda lengths and  $SL_2$ -tilings* (2023), available at <https://arxiv.org/abs/2306.17118>.
- [13] A. Hatcher, *Hyperbolic structures of arithmetic type on some link complements*, J. London Math. Soc. (2) **27** (1983), no. 2, 345–355.
- [14] ———, *Topology of numbers*, American Mathematical Society, Providence, RI, 2022.
- [15] T. Holm and P. Jørgensen, *A  $p$ -angulated generalisation of Conway and Coxeter’s theorem on frieze patterns*, Int. Math. Res. Not. IMRN **1** (2020), 71–90.
- [16] I. Ivrišimtzis and D. Singerman, *Regular maps and principal congruence subgroups of Hecke groups*, European J. Combin. **26** (2005), no. 3-4, 437–456.
- [17] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies, vol. No. 72, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1971.
- [18] S. Morier-Genoud, *Coxeter’s frieze patterns at the crossroads of algebra, geometry and combinatorics*, Bull. Lond. Math. Soc. **47** (2015), no. 6, 895–938.
- [19] ———, *Counting Coxeter’s friezes over a finite field via moduli spaces*, Algebr. Comb. **4** (2021), no. 2, 225–240.
- [20] S. Morier-Genoud, V. Ovsienko, and S. Tabachnikov,  *$SL_2(\mathbb{Z})$ -tilings of the torus, Coxeter-Conway friezes and Farey triangulations*, Enseign. Math. **61** (2015), no. 1-2, 71–92.
- [21] I. Short, *Classifying  $SL_2$ -tilings*, Trans. Amer. Math. Soc. **376** (2023), no. 1, 1–38.
- [22] D. Singerman and J. Strudwick, *The Farey maps modulo  $n$* , Acta Math. Univ. Comenian. (N.S.) **89** (2020), no. 1, 39–52.
- [23] L. Ya. Vulakh, *Farey polytopes and continued fractions associated with discrete hyperbolic groups*, Trans. Amer. Math. Soc. **351** (1999), no. 6, 2295–2323.

School of Mathematics and Statistics, The Open University,  
Milton Keynes, MK7 6AA, United Kingdom  
*E-mail address:* `ian.short@open.ac.uk`